Compact Hausdorff modal algebras are image-finite Kripke frames

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Outline

Image-finite frames I

An image-finite frame is a Kripke frame $\mathfrak{F} = \langle W, R \rangle$ such that for all $x \in W$, $\{y \mid xRy\}$ is finite.

Fact (Hennesy & Milner)

The class of image-finite frames has the Hennesy-Milner property with respect to the basic modal language.

Fact (Folklore?)

If $\mathfrak{F}$ is an image-finite frame, then the natural map $\mathfrak{F} \to \mathfrak{ue}\,\mathfrak{F}$ to the ultrafilter extension is a p-morphic embedding.

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Topological modal algebras

A structure $\mathbb{A} = \langle A; \wedge, \vee, \neg, 0, 1 \Diamond; \tau \rangle$ is a topological modal algebra if

- $\langle A; \wedge, \vee, \neg, 0, 1, \Diamond \rangle$ is a modal algebra,
- $\langle A; \tau \rangle$ is a topological space,
- every operation is continuous w.r.t. (the product topology of) τ.

By **KHausMA** we denote the category of compact Hausdorff topological modal algebras and continuous homomorphisms.

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Topological algebra compactification

Category theory tells us that there exist

- a functor $\beta\colon \mathbf{MA} \to \mathbf{KHausMA}$,
- a natural transformation $\eta\colon \mathrm{Id}_{\mathbf{MA}} \dot{\to} \beta$,

such that for every modal algebra $\mathbb{A}$, and for every compact Hausdorff modal algebra $\langle \mathbb{B}, \tau_{\mathbb{B}} \rangle$ and every homomorphism $f\colon \mathbb{A} \to \mathbb{B}$, there is a unique $f'\colon \beta\mathbb{A} \to \langle \mathbb{B}, \tau_{\mathbb{B}} \rangle$ such that the following diagram commutes.

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$$\mathbb{A} \xrightarrow{f} \langle \mathbb{B}, \tau_{\mathbb{B}} \rangle$$

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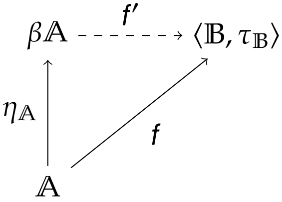

Example: compact Hausdorff Boolean algebras

- **KHausBA**: compact Hausdorff Boolean algebras and continuous homomorphisms.
- **CABA**: complete atomic Boolean algebras and complete homomorphisms.

Fact (Strauss)

KHausBA *is isomorphic to* **CABA**.

$$\beta : \mathbf{BA} \to \mathbf{KHausBA}$$
$$\mathbb{A} \mapsto \langle \mathcal{P}(\mathrm{Uf}\,\mathbb{A}), \tau_{\mathbb{A}} \rangle$$

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Image-finite frames II

Let $\mathfrak{F} = \langle W, R \rangle$ be a Kripke frame. (We allow $W = \emptyset$.) Recall that $\mathfrak{F}$ is image-finite if for all $x \in \mathfrak{F}$, $\{y \mid xRy\}$ is finite.

$$I_\omega \mathfrak{F} := \text{ largest image-finite generated subframe of } \mathfrak{F}.$$

Observe that I_ω is a functor: if $f \colon \mathfrak{F} \to \mathfrak{G}$ is a p-morphism and $x \in I_\omega \mathfrak{F}$, then $f(x) \in I_\omega \mathfrak{G}$.

- $\mathbf{I_\omega KFr}$: category of image-finite frames and p-morphisms.

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Dualities for Kripke frames: a reminder

Let $\mathbb{A}$ be a modal algebra.

- $\mathbb{A}_\bullet := \langle \mathrm{Uf}\,\mathbb{A}, R_\bullet^{\mathbb{A}} \rangle$ is the ultrafilter frame of $\mathbb{A}$. (Goldblatt)

Assume that $\mathbb{A}$ is also complete, atomic and completely additive.

- $\mathbb{A}_+ := \langle \mathrm{At}\,\mathbb{A}, R_+^{\mathbb{A}} \rangle$ is the discrete dual frame of $\mathbb{A}$.

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- $\mathfrak{F}^+ := \langle \mathcal{P}(W), \cap, \cup, (\cdot)^c, \emptyset, W, m_R \rangle$ is the complex algebra of $\mathfrak{F}$.

Duality: $\mathbb{A} \simeq (\mathbb{A}_+)^+$ and $\mathfrak{F} \simeq (\mathfrak{F}^+)_+$. (Thomason)
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A Representation Theorem

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***KHausMA** is dually equivalent to $\mathbf{I}_\omega$**KFr**.*

Recall: β: **MA** $\to$ **KHausMA** is the topological algebra compactification functor for modal algebras.

Theorem

Let $\mathbb{A}$ be a modal algebra. Then $\beta\mathbb{A}$ is (isomorphic to) the complex algebra of $I_\omega\mathbb{A}_\bullet$.

Remark: the natural map $\eta_\mathbb{A} : \mathbb{A} \to \beta\mathbb{A}$ is in general not injective.

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Duality applied: a toy example

Let $\mathfrak{F}$ be an image-finite frame and let $\mathbb{A} := \mathfrak{F}^+$. Consider $id_{\mathbb{A}} : \mathbb{A} \to \mathbb{A}$. Then

By the previous slide, dually this means that

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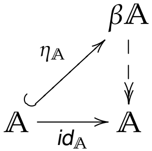

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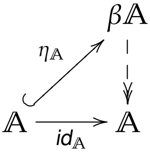

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