Axiomatizing ∇

M. Bílková & C. Kupke & A. Kurz & A. Palmigiano & Y. Venema

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Two Papers:

- Proof systems for the coalgebraic cover modality
  Marta Bílková, Alessandra Palmigiano & Yde Venema
- Completeness for the finitary Moss logic
  Clemens Kupke, Alexander Kurz & Yde Venema

Three talks:

- Yde Venema: Introduction
- Marta Bílková: Proof systems for the coalgebraic cover modality
- Alexander Kurz: Completeness for the finitary Moss logic

Overview of introduction

- The cover modality ∇
- Coalgebra
- Axiomatizations

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∇ as abbreviation

$$\nabla\alpha \equiv \Box \bigvee \alpha \wedge \bigwedge \Diamond\alpha$$

Reconstructing modal logic

Conversely

$$\begin{array}{lcl} \Diamond a & \equiv & \nabla\{a, \top\} \\ \Box a & \equiv & \nabla\varnothing \vee \nabla\{a\} \end{array}$$

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Proposition The languages ML and ML_∇ are effectively equi-expressive.

A modal distributive law

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Theorem For any sets α, α' of formulas,

$$\nabla\alpha \wedge \nabla\alpha' \equiv \bigvee_{\substack{Z \subseteq \alpha \times \alpha' \\ Z \text{ satisfies } \ldots}} \nabla\{a \wedge a' \mid (a, a') \in Z\}, \qquad \text{(MDL)}$$

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Then $\mathbb{S}, s \Vdash \nabla\{a \wedge a' \mid (a, a') \in Z_s\}$.

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But why???

There are many reasons not to switch to ML_∇ . . .

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- simulate alternating μ-automata into nondeterministic ones
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Also:

- connections with strategic normal form in evaluation games

Motivation 2

- ∇ provides new perspective on the Vietoris construction in topology (Palmigiano & Venema, 2007)

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Relation Lifting For $Z \subseteq S \times S'$, define $\overline{\mathsf{P}}(Z) \subseteq \mathsf{P}S \times \mathsf{P}S'$ by

$$\overline{\mathsf{P}}(Z) := \{(Q, Q') \in \mathsf{P}S \times \mathsf{P}S' \mid \forall q \in Q\, \exists q' \in Q'\, qZq' \;\&\; \forall q' \in Q'\, \exists q \in Q\, qZq' \}$$

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Corollary ML_{∇} can be generalized to coalgebras of arbitrary type!

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- A coalgebra is a structure $\mathbb{S} = \langle S, \sigma : S \to TS \rangle$,
  where T is the type of the coalgebra.
- Sufficiently general to model notions like:
  input, output, non-determinism, interaction, probability, . . .

Examples

- streams: $TS = C \times S$
- binary trees: $TS = C \times S \times S$
- Kripke frames: $TS = \mathsf{P}(S)$
- Kripke models: $TS = \mathsf{P}(\mathsf{Prop}) \times \mathsf{P}(S)$

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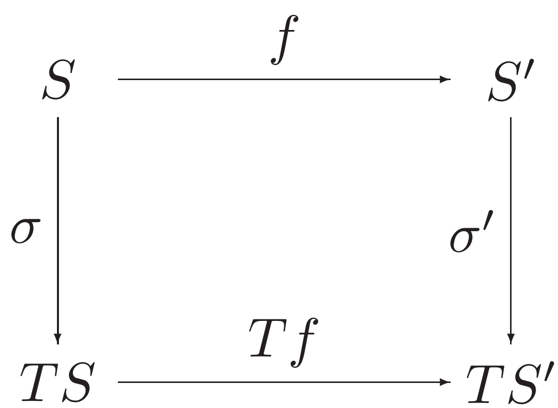
- An T-coalgebra is a pair $\mathbb{S} = \langle S, \sigma : S \to TS \rangle$.
- A coalgebra homomorphism between two coalgebras $\mathbb{S}$ and $\mathbb{S}'$ is a map $f : S \to S'$ such that $\sigma' \circ f = Tf \circ \sigma$:



Logics for Coalgebras

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Key Questions:

- What are good ‘coalgebraic’ languages for specifying behavior?
- What are good reasoning systems for these?
- . . .

Coalgebra and Modal Logic

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$$\frac{\text{Modal Logic}}{\text{Coalgebra}} = \frac{\text{Equational Logic}}{\text{Algebra}}$$

Implementations of this slogan:

- Generalize modal semantics to the coalgebraic level of abstraction:
  $\leadsto$ Moss' coalgebraic logic
- Look for Kripke structures 'inside coalgebras'
  $\leadsto$ predicate liftings provide (standard) modalities
- ...

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Inductively define Moss' language $\mathcal{L}_T$ with the following clause for ∇_T:

- if $\alpha \in T_\omega\mathcal{L}$ then $\nabla_T\alpha \in \mathcal{L}$

Here T_ω is the finitary version of T: $T_\omega S := \bigcup\{TS_0 \mid S_0 \subseteq_\omega S\}$.

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Categorical caveat For this to work well, T has to preserves weak pullbacks.

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Examples

(streams) if $T = \mathsf{Id}$ then $\nabla_T a = \mathsf{nexttime}(a)$
(binary trees) if $T = \mathsf{Id} \times \mathsf{Id}$ then $\nabla_T(a_0, a_1) = \langle \mathsf{left} \rangle a_0 \wedge \langle \mathsf{right} \rangle a_1$
(frames) if $T = \mathsf{P}$ then $\nabla_T = \nabla$
(models) if $TS = \mathsf{P}(\mathsf{Prop}) \times \mathsf{P}(S)$ then $\nabla_T(\Pi, \alpha) = \bigwedge_{p \in \Pi} p \wedge \bigwedge_{p \notin \Pi} \neg p \wedge \nabla \alpha$,
etc

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- Axiomatizations

First Axiomatization

Modal ∇-algebra: $\mathbb{A} = \langle A, \wedge, \vee, \top, \bot, \neg, \nabla \rangle$ with

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- ∇ satisfying ∇_1 – ∇_3 below:

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(∇_{2a}) $\nabla\alpha \wedge \nabla\beta \;\leq\; \bigvee\{\; \nabla\{a \wedge b \mid (a, b) \in Z\} \;\mid\; (\alpha, \beta) \in \overline{\mathsf{P}}(\leq) \;\}$,
(∇_{2b}) $\top \leq \nabla\varnothing \vee \nabla\{\top\}$,
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Corollary (Idem)
$\nabla_1 - \nabla_3$ form a complete axiomatization for ∇.

Carioca Axiomatization

$$\text{If } (\alpha, \beta) \in \overline{\mathsf{P}}(\leq), \text{ then } \nabla\alpha \leq \nabla\beta \tag{$\nabla 1$}$$

$$\bigwedge \left\{ \nabla\alpha \mid \alpha \in A \right\} \leq \bigvee_{\Phi \in SRD(A)} \nabla\{\bigwedge\beta \mid \beta \in \Phi\} \tag{$\nabla 2$}$$

$$\nabla\{\bigvee\alpha \mid \alpha \in \Phi\} \leq \bigvee_{(\beta,\Phi) \in \overline{\mathsf{P}}(\in)} \nabla\beta \tag{$\nabla 3$}$$

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$$\nabla\{\bigvee\alpha \mid \alpha \in \Phi\} \leq \bigvee_{(\beta,\Phi) \in \overline{\mathsf{P}}(\in)} \nabla\beta \tag{∇3}$$

Definition $\Phi \in \mathsf{P}_\omega\mathsf{P}_\omega(S)$ is a slim redistribution of $A \in \mathsf{P}_\omega\mathsf{P}_\omega(S)$ if

- $\bigcup B = \bigcup A$
- $\beta \cap \alpha \neq \varnothing$ for all $\beta \in B$ and all $\alpha \in A$

The set of slim redistributions of A is denoted as $SRD(A)$.

Note If $\in \subseteq S \times \mathsf{P}S$ then $\overline{\mathsf{P}}(\in) \subseteq \mathsf{P}S \times \mathsf{P}\mathsf{P}S$.

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$\leadsto$ Bílková's talk

Moss' Coalgebraic Language

- Our (finitary version of) Moss' language $\mathcal{L}_T$ is defined by

$$a \ ::= \ \bigvee \varphi \ \mid \ \bigwedge \varphi \ \mid \ \neg a \ \mid \ \nabla_T \alpha$$

where for convenience we fix notation as follows:

$\mathcal{L}$	$a, b, c, \ldots$
$T_\omega \mathcal{L}$	$\alpha, \beta, \gamma \ldots$
$\mathsf{P}_\omega \mathcal{L}$	$\varphi, \psi, \ldots$
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- Semantics: $\mathbb{S}, s \Vdash \nabla_T \alpha$ iff $(\sigma(s), \alpha) \in \overline{T}(\Vdash)$.

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  'there is a $Z \subseteq Sfor(\alpha) \times Sfor(\beta)$ such that $Z \subseteq \leq$ and $(\alpha, \beta) \in \overline{T}(Z)$'.
- Consider $\bigwedge$ as $\bigwedge : \mathsf{P}_\omega(\mathcal{L}) \to \mathcal{L}$, then $T\bigwedge : T_\omega\mathsf{P}_\omega(\mathcal{L}) \to T_\omega\mathcal{L}$.
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  Φ is slim, notation: $\Phi \in SRD(A)$, if Φ 'only uses material from A'.
- Categorically, define $\rho_S : T\mathsf{P}S \to \mathsf{P}TS$ by putting

$$\rho_S(\Phi) := \{\alpha \in TS \mid (\alpha, \Phi) \in \overline{T}(\in_S)\}.$$

  Then ρ is a natural transformation (a distributive law) if T preserves weak pullbacks.
  $\Phi \in SRD(A)$ iff $A = \rho_\mathcal{L}(\Phi)$.

Axioms for ∇_T

From $\alpha\overline{T}(\leq)\beta$ derive $\nabla_T\alpha \leq \nabla_T\beta$. $(\nabla 1)$

$$\bigwedge \left\{ \nabla_T\alpha \mid \alpha \in A \right\} \leq \bigvee_{\Phi \in SRD(A)} \nabla_T(T\bigwedge)(\Phi) \qquad (\nabla 2)$$

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Proof $\rightsquigarrow$ Kurz' talk.

This finishes the introduction . . .