Modal Logic of Time Division

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Outline

1. Background
2. Logic of Time Division
   - Syntax
   - Semantics
   - Basic examples
3. Expressive Power
   - Examples
   - Negative Properties
   - Positive Properties
4. Fragments

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von Wright's Approach to Tense Logic

- *Time, Change, and Contradiction* (1968).
- Change, duration.
- Focus on intervals and their division (instead of instants and their temporal succession).
- Approach remained informal: no formal semantics.
  - Dalla Chiara (1975/1989).
  - vW's notion of negation two-ways ambiguous.

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Syntax

- Syntactically, $\mathcal{L}_{TD}$ is **ML** with an extra unary operator ($\sim$):

$$\phi \ ::= \ p \mid \neg\phi \mid {\sim}\phi \mid (\phi \vee \phi) \mid (\phi \wedge \phi) \mid \Diamond\phi \mid \Box\phi$$

- Formulas of the forms p, ${\sim}p$, $\neg p$, $\neg{\sim}p$ are *literals*.

- Evaluation relative to intervals, consisting of instants.

- $\neg\phi$ says ϕ does not hold at the current interval.
- ${\sim}\phi$ says ϕ fails throughout the current interval.
- $\Box\phi$ says the current interval is divisible into (finitely many) subintervals each of which makes ϕ true ("chop-star").

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Occasions

- There are three kinds of **occasions** we consider:
  - intervals $]s, t]$ with $s < t$
  - singletons $\{t\}$
  - the empty occasion $\varnothing$

Left Bound, Interior Points

- Define the **left bound** of a non-empty occasion o:

$$l(o) := \begin{cases} \min(o) - 1 & \text{if } \min(o) - 1 \text{ exists} \\ \inf(o) & \text{otherwise} \end{cases}$$

  - If $o =]3,4] \subset \mathbb{Q}$: $l(o) = \inf(o) = 3$
  - If $o =]3,4] \subset \mathbb{N}$: $l(o) = \min(o) - 1 = 3$
  - If $o = \{4\} \subset \mathbb{Q}$: $l(o) = \inf(o) = 4$

- **Interior points** of o:

$$\mathbf{int}(o) := \begin{cases} \{t : l(o) < t < \max(o)\} & \text{if } o \text{ is not empty} \\ \varnothing & \text{otherwise} \end{cases}$$

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Divisions

- Suppose $t_1, \ldots, t_n$ are interior points of o, with $t_1 < \ldots < t_n$.

  The **division** of o by the points $t_1, \ldots, t_n$ is the partition

  $$\langle\ ]I(o), t_1],\ ]t_1, t_2],\ \ldots,\ ]t_n, \max(o)]\ \rangle,$$

  denoted $\mathbb{D}_o(t_1, \ldots, t_n)$.

  The members of the partition are its *cells*.

Semantics

- **Models** $\mathcal{M} = (T, <, V)$ usual modal structures, where $<$ is **linear** (irreflexive, transitive, trichotomous).
- Evaluation on models $\mathcal{M}$ at occasions o.

$$
\begin{array}{rcl}
\mathcal{M}, o \models p & \text{iff} & t \in V(p) \text{ for all } t \in o \\
\mathcal{M}, o \models \neg\psi & \text{iff} & \mathcal{M}, o \not\models \psi \\
\mathcal{M}, o \models \sim\psi & \text{iff} & \mathcal{M}, \{t\} \not\models \psi \text{ for all } t \in o \\
\mathcal{M}, o \models (\psi \wedge \chi) & \text{iff} & \mathcal{M}, o \models \psi \ \text{ and } \ \mathcal{M}, o \models \chi \\
\mathcal{M}, o \models (\psi \vee \chi) & \text{iff} & \mathcal{M}, o \models \psi \ \text{ or } \ \mathcal{M}, o \models \chi
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Semantics, cont.

$\mathcal{M}, o \models \Box\psi$ iff for some $n > 0$ there are successive points $t_1, \ldots, t_n$ interior to o such that for every $o' \in \mathbb{D}_o(t_1, \ldots, t_n)$, we have $\mathcal{M}, o' \models \psi$

$\mathcal{M}, o \models \Diamond\psi$ iff for all $n > 0$ and all successive points $t_1, \ldots, t_n$ interior to o, there is $o' \in \mathbb{D}_o(t_1, \ldots, t_n)$ such that $\mathcal{M}, o' \models \psi$

Basic Examples

- $\varnothing \models p$ and $\varnothing \models {\sim}\phi$ and $\varnothing \models \Diamond\psi$.
- $o \models {\sim}\top$ iff $o = \varnothing$.

- $o \models \neg p$ iff p fails at some $t \in o$.
- $o \models \neg{\sim}p$ iff p holds at some $t \in o$.

- Distribution of $\sim$ over $\wedge$ fails:

$$\sim(\sim p \wedge \sim q) \not\equiv (p \vee q).$$

Proof.

Suppose all and only rational (irrational) points in $]1,2]$ make p true (q true). Then $]1,2] \models \sim(\sim p \wedge \sim q)$ but $]1,2] \not\models (p \vee q)$. $\square$

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Basic Examples, Cont.

- $o \models \sim\Box\phi$.

- $o \models \Box\neg p$ iff p fails at least twice during o.
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Suppose $|o| \geq 2$.

- $o \models \Diamond\Box\neg p$ iff p fails infinitely often during o.

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Outline

Examples

- Expressing infinity: $o \models (\Box\top \wedge \Diamond\Box\top)$ iff $|o|$ is infinite.
- Varieties of indivisibility:

(a) $T = \mathbb{Q}$; $V(p)$ and $o \setminus V(p)$ are both dense in o.

$$]1,2] \not\models \Box(p \vee \sim p).$$

(b) $T = \mathbb{Q}$; $V(p) = \{t : t^2 < 2\}$.

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(c) $T = \mathbb{R}$; $V(p) = \{t : t^2 < 2\}$.

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Negative Properties

Fact

$\mathcal{L}_{TD}$ lacks the f.m.p.

Theorem

*$\mathcal{L}_{TD} \not\leq$ **FO**.*

Proof.

Let $n < \omega$ and consider models $\mathfrak{M}_n$ and $\mathfrak{M}$:

$$\mathfrak{M}_n \qquad 0 \longrightarrow 1 \longrightarrow \cdots \longrightarrow 2^n$$

$$\mathfrak{M} \qquad 0 \longrightarrow 1 \longrightarrow \cdots \longrightarrow n \longrightarrow \cdots \longrightarrow -m \longrightarrow \cdots \longrightarrow -2 \longrightarrow -1$$

- $\mathfrak{M}_n, \mathfrak{M}$ are distinguished by $(\Box\top \wedge \Diamond\Box\top)$.
- Structures $\langle \mathfrak{M}_n, 0, 2^n \rangle$ and $\langle \mathfrak{M}, 0, -1 \rangle$ are not distinguished by any **FO** formula $\phi(x, y)$ of quantifier rank $\leq n$. $\square$

Positive Properties

- $\mathcal{L}_{TD} \leq \mathcal{L}_{\mathrm{w}}^{\mathrm{mon}}$. Actually $\mathcal{L}_{TD} < \mathcal{L}_{\mathrm{w}}^{\mathrm{mon}}$.
- $\mathcal{L}_{TD}$ has the downwards Löwenheim-Skolem property: any satisfiable formula is true in a *countable* $(\mathcal{M}, o)$.
- $\mathcal{L}_{TD}$ is decidable (Läuchli 1968).

Outline

Negation Normal Form

- Level 0 formulas:
  p, $\sim p$ and their $\wedge$, $\vee$ combinations.
- Base formulas:
  $p, \sim p, \neg p, \neg\sim p$ as well as $\sim(\mathbf{a} \wedge \mathbf{b})$ and $\neg\sim(\mathbf{a} \wedge \mathbf{b})$, where $\mathbf{a}, \mathbf{b}$ are level 0 formulas.
- Write $\mathcal{L}_{nnf}$ for the result of closing base formulas under $\wedge, \vee, \Box, \Diamond$.

Fact (Negation normal form)

$\mathcal{L}_{TD} = \mathcal{L}_{nnf}$.

Prenex formulas

- Prenex formulas ($\mathcal{L}_{PR}$):

$$(\Diamond\Box)^n\ell, \quad (\Box\Diamond)^n\ell, \quad \Box(\Diamond\Box)^n\ell, \quad \Diamond(\Box\Diamond)^n\ell.$$

- Let $\mathcal{D}_{>1}$ be the set of all anchored models $(\mathcal{M}, o)$ with a dense linear order and $|o| > 1$.

Theorem (Decidability of the $\wedge, \vee$ closure of prenex formulas)

Let $\mathcal{L}_{BPR}$ be the $\wedge, \vee$ closure of $\mathcal{L}_{PR}$. Over $\mathcal{D}_{>1}$, the satisfiability and validity problems of $\mathcal{L}_{BPR}$ are **NP***-complete.*

Further fragments

- Propositional fragment:

  Base formulas and their $\wedge$, $\vee$ combinations.

- Simple fragment:

  $\mathbf{c}$, $\Box\mathbf{c}$, $\Diamond\mathbf{c}$ with $\mathbf{c}$ propositional and their $\wedge$, $\vee$ combinations.