Locality and Subsumption Testing

in $\mathcal{EL}$ and some of its extensions

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Motivation

Description logics – used for KR in databases/ontologies

Provide a logical basis for modeling and reasoning about:

- objects
- classes of objects (concepts)
- relationshops between objects (links, roles)

Wide variety of description logics (various degrees of expressivity)

This talk: Tractable description logic: $\mathcal{EL}, \mathcal{EL}^+$ and extensions
[Baader'03–] used e.g. in medical ontologies (SNOMED)

Motivation

Description logics – used for KR in databases/ontologies

Focus: Tractable description logic: $\mathcal{EL}, \mathcal{EL}^+$ and extensions [Baader'03–]
used e.g. in medical ontologies (SNOMED)

Main contributions of this paper:

- Alternative proof of tractability of $\mathcal{EL}, \mathcal{EL}^+$ based on a notion of locality
  - $\mapsto$ a hierarchical reduction to SAT in the case of $\mathcal{EL}$
  - $\mapsto$ reduction to SAT of ground Horn formulae in the case of $\mathcal{EL}^+$
- Identify tractable extensions of $\mathcal{EL}$ and $\mathcal{EL}^+$
  - $\mapsto$ with n-ary roles and/or numerical domains.

Overview

- **Introduce $\mathcal{EL}$ and $\mathcal{EL}^+$ and the deduction problems**
  (subsumption w.r.t. a TBox resp. CBox)

- **Algebraic semantics**

  TBox/CBox subsumption $\mapsto$ u.w.p. for $\mathsf{SLat} \cup \mathsf{Mon}(\Sigma)(\cup \mathsf{Ax})$

- **Local theories and local theory extensions**

- **Locality results for** $\mathsf{SLat} \cup \mathsf{Mon}(\Sigma)(\cup \mathsf{Ax})$

- **Tractable extensions of $\mathcal{EL}, \mathcal{EL}^+$**

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$\mathcal{EL}$: Generalities

Concepts: • primitive concepts N_C

• complex concepts (built using concept constructors $\sqcap$, $\exists r$)

Roles: N_R

Interpretations: $\mathcal{I} = (D^{\mathcal{I}}, \cdot^{\mathcal{I}})$

• $C \in N_C \mapsto C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$

• $r \in N_R \mapsto r^{\mathcal{I}} \subseteq D^{\mathcal{I}} \times D^{\mathcal{I}}$

Constructor name	Syntax	Semantics
conjunction	$C_1 \sqcap C_2$	$C_1^{\mathcal{I}} \cap C_2^{\mathcal{I}}$
existential restriction	$\exists r.C$	$\{x \mid \exists y((x, y) \in r^{\mathcal{I}} \text{ and } y \in C^{\mathcal{I}})\}$

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- $r \in N_R \mapsto r^{\mathcal{I}} \subseteq D^{\mathcal{I}} \times D^{\mathcal{I}}$

Problem:

Given: TBox (set $\mathcal{T}$ of concept inclusions $C_i \sqsubseteq D_i$) and concepts C, D

Task: test whether $C \sqsubseteq_{\mathcal{T}} D$,

i.e. whether for all $\mathcal{I}$ if $C_i^{\mathcal{I}} \subseteq D_i^{\mathcal{I}}$ $\forall C_i \sqsubseteq D_i \in \mathcal{T}$ then $C^{\mathcal{I}} \subseteq D^{\mathcal{I}}$

Decidable in PTIME [Baader02] (graphs/simulation, transl. to Datalog, ...)

$\mathcal{EL}$: Example

Primitive concepts:	protein, process, substance
Roles:	catalyzes, produces
Terminology:	$\text{enzyme} = \text{protein} \sqcap \exists \text{catalyzes}.\text{reaction}$
(TBox)	$\text{catalyzer} = \exists \text{catalyzes}.\text{process}$
	$\text{reaction} = \text{process} \sqcap \exists \text{produces}.\text{substance}$
Query:	$\text{enzyme} \sqsubseteq \text{catalyzer}$?

Algebraic semantics for $\mathcal{EL}$

Translation of concept descriptions to terms:

$$\begin{array}{ccl} C & \mapsto & \overline{C} \text{ for any concept name } C \\ C_1 \sqcap C_2 & \mapsto & \overline{C_1 \sqcap C_2} = \overline{C}_1 \wedge \overline{C}_2 \\ \exists r.C & \mapsto & \overline{\exists r.C} = f_r(\overline{C}) \end{array}$$

Algebraic semantics for $\mathcal{EL}$.

Assume that $N_C = \{c_1, \dots, c_n\}$. The following are equivalent:

(1) $C \sqsubseteq_{\mathcal{T}} D$

(2) $\text{BAO} \cup \text{Jh}(\{f_r \mid r \in N_R\}) \models \forall c_1, \dots, c_n((\bigwedge_{C_i \sqsubseteq D_i \in \mathcal{T}} \overline{C}_i \leq \overline{D}_i) \rightarrow \overline{C} \leq \overline{D})$

(3) $\text{SLat} \cup \text{Mon}(\{f_r \mid r \in N_R\}) \models \forall c_1, \dots, c_n((\bigwedge_{C_i \sqsubseteq D_i \in \mathcal{T}} \overline{C}_i \leq \overline{D}_i) \rightarrow \overline{C} \leq \overline{D})$

Algebraic semantics for $\mathcal{EL}$

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Primitive concepts:	protein, process, substance
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Terminology:	enzyme $=$ protein $\sqcap$ $\exists$catalyzes.reaction
(TBox)	catalyzer $=$ $\exists$catalyzes.process
	reaction $=$ process $\sqcap$ $\exists$produces.substance
Query:	enzyme $\sqsubseteq$ catalyzer?

$$\begin{aligned}\text{SLat} \cup \text{Mon} \models\ & \text{enzyme} = \text{protein} \sqcap \text{catalyzes-some}(\text{reaction}) \quad \wedge \\ & \text{catalyzer} = \text{catalyze-some}(\text{process}) \quad \wedge \\ & \text{reaction} = \text{process} \sqcap \text{produces-some}(\text{substance}) \\ & \Rightarrow \text{enzyme} \sqsubseteq \text{catalyzer}\end{aligned}$$

$$\begin{aligned}\text{Mon}: \ & \forall C, D (C \sqsubseteq D \rightarrow \text{catalyze-some}(C) \sqsubseteq \text{catalyze-some}(D)) \\ & \forall C, D (C \sqsubseteq D \rightarrow \text{produces-some}(C) \sqsubseteq \text{produces-some}(D))\end{aligned}$$

$\mathcal{EL}^+$: generalities

Concepts:
- primitive concepts N_C
- complex concepts (built using concept constructors $\sqcap, \exists r$)

Roles: N_R

Interpretations: $\mathcal{I} = (D^\mathcal{I}, \cdot^\mathcal{I})$
- $C \in N_C \mapsto C^\mathcal{I} \subseteq D^\mathcal{I}$
- $r \in N_R \mapsto r^\mathcal{I} \subseteq D^\mathcal{I} \times D^\mathcal{I}$

Problem:

Given: CBox $\mathcal{C} = (\mathcal{T}, RI)$, where $\mathcal{T}$ set of concept inclusions $C_i \sqsubseteq D_i$;
RI set of role inclusions $r \circ s \sqsubseteq t$ or $r \sqsubseteq t$
concepts C, D

Task: test whether $C \sqsubseteq_\mathcal{T} D$,
i.e. whether for all $\mathcal{I}$ if $\underset{\forall C_i \sqsubseteq D_i \in \mathcal{T}}{C_i^\mathcal{I} \subseteq D_i^\mathcal{I}}$ and $\underset{\forall r \circ s \sqsubseteq t \in RI}{r^\mathcal{I} \circ s^\mathcal{I} \subseteq t^\mathcal{I}}$ then $C^\mathcal{I} \subseteq D^\mathcal{I}$

$\mathcal{EL}^+$: Example

Primitive concepts:	protein, process, substance
Roles:	catalyzes, produces, helps-producing
Terminology:	enzyme $=$ protein $\sqcap$ $\exists$catalyzes.reaction
(TBox)	reaction $=$ process $\sqcap$ $\exists$produces.substance
Role inclusions:	catalyzes $\circ$ produces $\sqsubseteq$ helps-producing
Query:	enzyme $\sqsubseteq$ protein $\sqcap$ $\exists$helps-producing.substance ?

Algebraic semantics for $\mathcal{EL}^+$

Translation of concept descriptions to terms:

$$\begin{array}{ccl} C & \mapsto & \overline{C} \text{ for any concept name } C \\ C_1 \sqcap C_2 & \mapsto & \overline{C_1 \sqcap C_2} = \overline{C}_1 \wedge \overline{C}_2 \\ \exists r.C & \mapsto & \overline{\exists r.C} = f_r(\overline{C}) \end{array}$$

$$RI \quad \mapsto \quad Ax(RI) = \{\forall x\, f_r(f_s(x)) \leq f_t(x) \mid r \circ s \subseteq t \in RI\}$$

Algebraic semantics for $\mathcal{EL}^+$

Assume that $N_C = \{c_1, \dots, c_n\}$. T.f.a.e. for any $\mathcal{EL}^+ C$ Box $\mathcal{C} = (\mathcal{T}, RI)$

(1) $C \sqsubseteq_{\mathcal{C}} D$

(2) $\mathsf{SLat} \cup \mathsf{Mon}(\{f_r | r \in N_R\}) \cup Ax(RI) \models \forall c_1, \dots, c_n((\bigwedge_{C_i \sqsubseteq D_i \in \mathcal{T}} \overline{C}_i \leq \overline{D}_i) \rightarrow \overline{C} \leq \overline{D})$

$\mathcal{EL}^+$: Algebraic semantics

Primitive concepts:	protein, process, substance
Roles:	catalyzes, produces, helps-producing
Terminology: (TBox)	enzyme $=$ protein $\sqcap$ $\exists$catalyzes.reaction reaction $=$ process $\sqcap$ $\exists$produces.substance
Role inclusions:	catalyzes $\circ$ produces $\sqsubseteq$ helps-producing
Query:	enzyme $\sqsubseteq$ protein $\sqcap$ $\exists$helps-producing.substance ?

$$
\begin{aligned}
\text{SLat} \cup \text{Mon} \cup Ax(RI) \models\; & \text{enzyme} = \text{protein} \sqcap \text{catalyzes-some}(\text{reaction}) \wedge \\
& \text{reaction} = \text{process} \sqcap \text{produces-some}(\text{substance}) \\
& \Rightarrow \text{enzyme} \sqsubseteq \text{protein} \sqcap \text{helps-producing-some}(\text{substance})
\end{aligned}
$$

$$
\begin{aligned}
\text{Mon}: \;& \forall C, D(C \sqsubseteq D \rightarrow \text{catalyze-some}(C) \sqsubseteq \text{catalyze-some}(D)) \\
& \forall C, D(C \sqsubseteq D \rightarrow \text{produces-some}(C) \sqsubseteq \text{produces-some}(D))
\end{aligned}
$$

$$Ax(RI) : \forall x(\text{catalyzes-some}(\text{produces-some}(x)) \leq \text{helps-producing}(x))$$

Efficient reasoning in the algebraic models

- TBox subsumption in $\mathcal{EL} \mapsto$ uniform word problem w.r.t. $\mathsf{SLat} \cup \mathsf{Mon}(\Sigma)$
- CBox subsumption in $\mathcal{EL}^+ \mapsto$ uniform word problem $\mathsf{SLat} \cup \mathsf{Mon}(\Sigma) \cup Ax$,

  where Ax consists of axioms of the type:

  $\forall x(f(x) \leq g(x))$ and $\forall x(f(g(x)) \leq h(x))$

The uniform word problem for $SLat \cup \mathsf{Mon}(\Sigma)(\cup Ax)$: decidable in PTIME

Explanation: Local theories / local theory extensions

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Local theories

Local theories [McAllester & Givan'92; Ganzinger'01]

$\mathcal{K}$ set of equational Horn clauses

$\mathcal{K}$ is local, if for ground clauses G,

$$\mathcal{K} \cup G \models \perp \text{ iff } \mathcal{K}[G] \cup G \models \perp$$

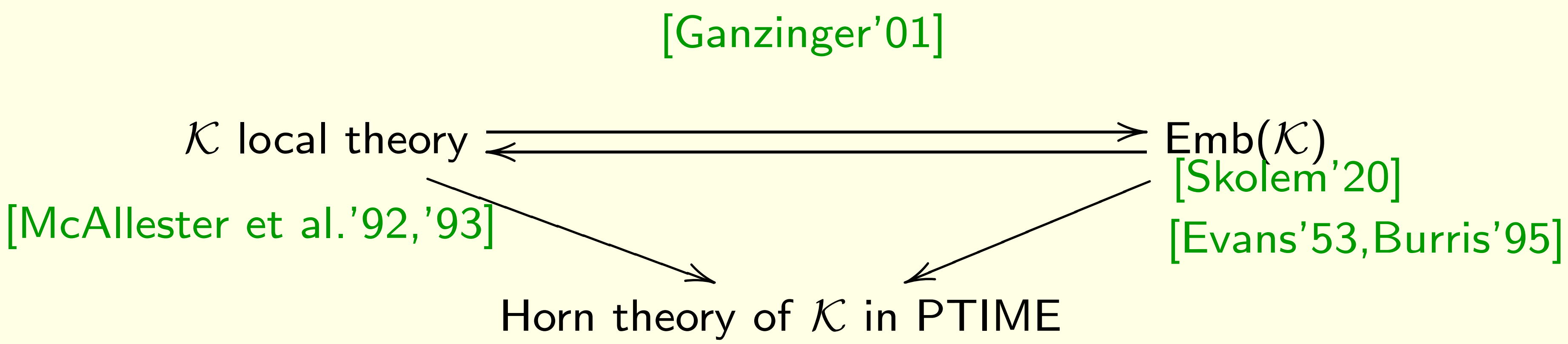
[McAllester, Givan '92, '93] Local theories capture PTIME

Local theories

Local theories [McAllester & Givan'92; Ganzinger'01]

$\mathcal{K}$ set of equational Horn clauses

> $\mathcal{K}$ is **local**, if for ground clauses G,
> $\mathcal{K} \cup G \models \perp$ iff $\mathcal{K}[G] \cup G \models \perp$



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Example:

- Axiomatization of lattices [Skolem 1920]

  Locality: every poset embeds into a lattice (Dedekind-MacNeille completion)

- Similar results for axiomatizations of semilattices

- Several other examples (algebra [Burris95]; verification: theories of lists ...)

Local theories

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$\mathcal{K} \cup G \models \perp$ iff $\mathcal{K}[G] \cup G \models \perp$

Compl: Pol. in size of G
Ex: $G : f(c)=f(d)$ $\mathrm{Mon}_f[G] : c \leq d \rightarrow f(c) \leq f(d)$

Ψ closure operation on ground terms.

$\mathcal{K}$ is Ψ-local, if for ground clauses G,

$\mathcal{K} \cup G \models \perp$ iff $\mathcal{K}[\Psi(G)] \cup G \models \perp$

Compl: Pol. in size of $\Psi(G)$

$\mathcal{T}_0 \subseteq \mathcal{K}$ is stably local, if for ground clauses G,

$\mathcal{T}_0 \cup \mathcal{K} \cup G \models \perp$ iff $\mathcal{T}_0 \cup \mathcal{K}^{[G]} \cup G \models \perp$

Compl: Pol. in size of G
Ex: $G : f(c)=f(d)$ $Mon_f^{[G]}: c \leq d \rightarrow f(c) \leq f(d)$ $f(c) \leq d \rightarrow f(f(c)) \leq f(d)$

Local theory extensions

Local theory extensions [Ganzinger,VS,Waldmann'04, VS'05]

$\mathcal{K}$ set of equational clauses; $\mathcal{T}_0$ theory; $\mathcal{T}_1 = \mathcal{T}_0 \cup \mathcal{K}$

> $\mathcal{T}_0 \subseteq \mathcal{K}$ is local, if for ground clauses G,
>
> $$\mathcal{T}_0 \cup \mathcal{K} \cup G \models \perp \text{ iff } \mathcal{T}_0 \cup \mathcal{K}[G] \cup G \models \perp$$

Ψ closure operation on ground terms. [Ihlemann, Jacobs. VS'08]

> $\mathcal{T}_0 \subseteq \mathcal{K}$ is Ψ-local, if for ground clauses G,
>
> $$\mathcal{T}_0 \cup \mathcal{K} \cup G \models \perp \text{ iff } \mathcal{T}_0 \cup \mathcal{K}[\Psi(G)] \cup G \models \perp$$

Stably local theory extensions are defined similarly.

Reasoning in local theory extensions

Locality: $\mathcal{T}_0 \cup \mathcal{K} \cup G \models \perp$ iff $\mathcal{T}_0 \cup \mathcal{K}[\Psi(G)] \cup G \models \perp$

Problem: Decide whether $\mathcal{T}_0 \cup \mathcal{K}[\Psi(G)] \cup G \models \perp$

Solutions:

1: Use $SMT(\mathcal{T}_0 + UIF)$: completeness guaranteed only if $\mathcal{K}[\Psi(G)]$ ground

2: Hierarchical reasoning [VS'05]

reduce to satisfiability in $\mathcal{T}_0$: applicable in general; sound and complete

$\mapsto$ parameterized complexity

Locality and $\mathcal{EL}, \mathcal{EL}^+$

Problem: Determine if a theory (or a theory extension) is local

Solution: Embeddability of partial models into total models

$$\mathcal{T}_1 \text{ } \Psi\text{-local extension of } \mathcal{T}_0 \rightleftarrows \text{Emb}(\mathcal{T}_0, \mathcal{T}_1)$$

extends [Ganzinger'01, GSW'04, VS'05, VS,Ihlemann'07, Ihlemann,Jacobs,VS'08]

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Problem: Determine if a theory (or a theory extension) is local

Solution: Embeddability of partial models into total models

$$\mathcal{T}_1 \ \Psi\text{-local extension of } \mathcal{T}_0 \Longleftrightarrow \mathsf{Emb}(\mathcal{T}_0, \mathcal{T}_1)$$

extends [Ganzinger'01, GSW'04, VS'05, VS,Ihlemann'07, Ihlemann,Jacobs,VS'08]

This criterion provides a large number of examples:

- Mathematics: ext. with monotone functions; Lipschitz functions, ...
- Verification: Theories of pointer structures, lists, records, arrays.
- Kryptography: Encode/decode with given key

Locality and $\mathcal{EL}, \mathcal{EL}^+$

Problem: Determine if a theory (or a theory extension) is local

Solution: Embeddability of partial models into total models

$\mathcal{T}_1$ Ψ-local extension of $\mathcal{T}_0$ $\rightleftarrows$ $\text{Emb}(\mathcal{T}_0, \mathcal{T}_1)$

extends [Ganzinger'01, GSW'04, VS'05, VS,Ihlemann'07, Ihlemann,Jacobs, VS'08]

- P is a weak partial model of a Horn clause $C = \bigwedge L_i \rightarrow L$ if
  for each valuation β, if all terms in C are defined then C is true in P.

- P is an Evans partial model of a Horn clause $C = \bigwedge L_i \rightarrow s \rho t$ if
  for each valuation β, if all terms in the premises are defined then
  - either all terms in L are defined and L true in P
  - if $s{=}f(s')$ and s', t defined then $f(s')$ defined in P and $f(s')\rho t$ true
  - s' is undefined, or s and t are undefined.

Locality and $\mathcal{EL}, \mathcal{EL}^+$

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$$\mathcal{T}_1 \text{ } \Psi\text{-local extension of } \mathcal{T}_0 \rightleftarrows \mathsf{Emb}(\mathcal{T}_0, \mathcal{T}_1)$$

extends [Ganzinger'01, GSW'04, VS'05, VS,Ihlemann'07, Ihlemann,Jacobs,VS'08]

Examples of local theory extensions related to $\mathcal{EL}$:

Fact: Every semilattice with partial functions, monotone on their (finite) domain of definition (weakly) embeds into a semilattice with total monotone functions.

Consequence: $\mathsf{SLat} \subseteq \mathsf{SLat} \cup \mathsf{Mon}(\Sigma)$ is a local theory extension

Locality and $\mathcal{EL}, \mathcal{EL}^+$

Problem: Determine if a theory (or a theory extension) is local

Solution: Embeddability of partial models into total models

$$\mathcal{T}_1 \ \Psi\text{-local extension of } \mathcal{T}_0 \ \Longleftrightarrow \ \mathsf{Emb}(\mathcal{T}_0, \mathcal{T}_1)$$

extends [Ganzinger'01, GSW'04, VS'05, VS,Ihlemann'07, Ihlemann,Jacobs,VS'08]

Examples of local theories related to $\mathcal{EL}$, $\mathcal{EL}^+$:

There exists a presentation SL of the theory of semilattices such that:

Fact: Every weak partial model of $SL \cup \mathsf{Mon}(\Sigma)$ (weakly) embeds into a total one
Consequence: $SL \cup \mathsf{Mon}(\Sigma)$ is a local theory

Fact: Every Evans partial model A of $SL \cup \mathsf{Mon}(\Sigma) \cup Ax$ such that:
for every $f(g(x)) \leq h(x) \in \mathsf{Ax}$ if $h_A(a)$ is defined then $g_A(a)$ is defined.
Consequence: $SL \cup \mathsf{Mon}(\Sigma) \cup Ax(RI)$ is Ψ-stably local,
(Ψ adds $f_s(u)$ to the set of terms whenever $f_t(u)$ is in it and $r \circ s \subseteq t \in RI$)

$\mathcal{EL}$: Hierarchical reasoning

Primitive concepts:	protein, process, substance
Roles:	catalyzes, produces
Terminology:	enzyme $=$ protein $\sqcap$ $\exists$catalyzes.reaction
(TBox)	catalyzer $=$ $\exists$catalyzes.process
	reaction $=$ process $\sqcap$ $\exists$produces.substance
Query:	enzyme $\sqsubseteq$ catalyzer?

$$
\begin{aligned}
\text{SLat} \cup \text{Mon} \models\, & \text{enzyme} = \text{protein} \sqcap \text{catalyzes-some}(\text{reaction}) \quad \wedge \\
& \text{catalyzer} = \text{catalyze-some}(\text{process}) \quad \wedge \\
& \text{reaction} = \text{process} \sqcap \text{produces-some}(\text{substance}) \\
& \Rightarrow \text{enzyme} \sqsubseteq \text{catalyzer}
\end{aligned}
$$

$$
\begin{aligned}
\text{Mon}: \quad & \forall C, D (C \sqsubseteq D \rightarrow \text{catalyze-some}(C) \sqsubseteq \text{catalyze-some}(D)) \\
& \forall C, D (C \sqsubseteq D \rightarrow \text{produces-some}(C) \sqsubseteq \text{produces-some}(D))
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$\mathcal{EL}$: Hierarchical reasoning

$$\mathsf{SLat} \cup \mathsf{Mon} \wedge \underbrace{\begin{array}{l} \mathsf{enzyme} = \mathsf{protein} \sqcap \mathsf{catalyzes\text{-}some}(\mathsf{reaction}) \wedge \\ \mathsf{catalyzer} = \mathsf{catalyze\text{-}some}(\mathsf{process}) \wedge \\ \mathsf{reaction} = \mathsf{process} \sqcap \mathsf{produces\text{-}some}(\mathsf{substance}) \wedge \\ \mathsf{enzyme} \not\sqsubseteq \mathsf{catalyzer} \end{array}}_{G} \models \perp$$

$G \wedge \mathsf{Mon}$
$\mathsf{enzyme} = \mathsf{protein} \sqcap \mathsf{catalyzes\text{-}some}(\mathsf{reaction}) \wedge$ $\mathsf{catalyzer} = \mathsf{catalyze\text{-}some}(\mathsf{process}) \wedge$ $\mathsf{reaction} = \mathsf{process} \sqcap \mathsf{produces\text{-}some}(\mathsf{substance}) \wedge$ $\mathsf{enzyme} \not\sqsubseteq \mathsf{catalyzer}$
$\forall C, D (C \sqsubseteq D \rightarrow \mathsf{catalyze\text{-}some}(C) \sqsubseteq \mathsf{catalyze\text{-}some}(D))$ $\forall C, D (C \sqsubseteq D \rightarrow \mathsf{produces\text{-}some}(C) \sqsubseteq \mathsf{produces\text{-}some}(D))$

$\mathcal{EL}$: Hierarchical reasoning

$$\text{SLat} \cup \text{Mon} \wedge \underbrace{\begin{array}{l} \text{enzyme} = \text{protein} \sqcap \text{catalyzes-some}(\text{reaction}) \wedge \\ \text{catalyzer} = \text{catalyze-some}(\text{process}) \wedge \\ \text{reaction} = \text{process} \sqcap \text{produces-some}(\text{substance}) \wedge \\ \text{enzyme} \not\leq \text{catalyzer} \end{array}}_{G} \models \perp$$

Solution 1: Use $DPLL(\text{SLat} + UIF)$

$G \wedge \text{Mon}[G]$
$\text{enzyme} = \text{protein} \sqcap \text{catalyzes-some}(\text{reaction})$ $\text{catalyzer} = \text{catalyzes-some}(\text{process})$ $\text{reaction} = \text{process} \sqcap \text{produces-some}(\text{substance})$ $\text{enzyme} \not\leq \text{catalyzer}$
$\text{reaction} \rhd \text{process} \rightarrow \text{catalyzes-some}(\text{reaction}) \rhd \text{catalyzes-some}(\text{process}), \ \rhd \in \{\leq, \geq, =\}$

$\mathcal{EL}$: Hierarchical reasoning

$$\text{SLat} \cup \text{Mon} \wedge \underbrace{\begin{array}{l} \text{enzyme} = \text{protein} \sqcap \text{catalyzes-some}(\text{reaction}) \wedge \\ \text{catalyzer} = \text{catalyze-some}(\text{process}) \wedge \\ \text{reaction} = \text{process} \sqcap \text{produces-some}(\text{substance}) \wedge \\ \text{enzyme} \not\leq \text{catalyzer} \end{array}}_{G} \models \perp$$

Solution 2: Hierarchical reasoning

Base theory (SLat)	Extension
$\text{enzyme} = \text{protein} \sqcap c_1$	$c_1 = \text{catalyzes-some}(\text{reaction})$
$\text{catalyzer} = c_2$	$c_2 = \text{catalyzes-some}(\text{process})$
$\text{reaction} = \text{process} \sqcap c_3$	$c_3 = \text{produces-some}(\text{substance})$
$\text{enzyme} \not\leq \text{catalyzer}$	
$\text{reaction} \rhd \text{process} \rightarrow c_1 \rhd c_2 \quad \rhd \in \{\leq, \geq, =\}$	

Test satisfiability using any prover for SLat (e.g. reduction to SAT)

$\mathcal{EL}^+$: Hierarchical reasoning

- Ψ-stable locality result

  $\Psi(G) = \text{st}(G) \cup \{f_2(d) \mid f(d) \in \text{st}(G), f_1(f_2(x)) \leq f(x) \in Ax(RI)\} \cup \ldots$

  $\mapsto$ need to take into account more ground terms

Example:

st(G) = {enzyme, protein, reaction, catalyzer, substance
catalyzes-some(reaction), helps-producing(substance) }

Ax(RI): catalyzes-some(produces-some(x)) $\leq$ helps-producing(x)

$\mathcal{EL}^+$: Hierarchical reasoning

- Ψ-stable locality result

  $\Psi(G) = \text{st}(G) \cup \{f_2(d) \mid f(d) \in \text{st}(G), f_1(f_2(x)) \leq f(x) \in Ax(RI)\} \cup \dots$

  $\mapsto$ need to take into account more ground terms

Example:

st(G) = {enzyme, protein, reaction, catalyzer, substance

catalyzes-some(reaction), helps-producing(substance) }

Ax(RI): catalyzes-some(produces-some(x)) $\leq$ helps-producing(x)

helps-producing(substance) $\in$ st(G) $\Rightarrow$ produces-some(substance) $\in \Psi(G)$

... and (by stable locality) more instances

$\mapsto$ all instances of $SL \cup \text{Mon} \cup Ax(RI)$ with subst. with codomain $\Psi(G)$

- still in PTIME

Other examples of local theories

Similar locality results also for n-ary monotone functions:

- $\mathsf{SLat} \subseteq \mathsf{SLat} \cup \mathsf{Mon}(\Sigma)$ is a local theory extension
- There exists a presentation SL of the theory of semilattices such that $SL \cup \mathsf{Mon}(\Sigma)$ is a local theory

Other examples of local theories

Similar locality results also for *n*-ary monotone functions:

- $\mathsf{SLat} \subseteq \mathsf{SLat} \cup \mathsf{Mon}(\Sigma)$ is a local theory extension
- There exists a presentation SL of the theory of semilattices such that $SL \cup \mathsf{Mon}(\Sigma)$ is a local theory

... and for many-sorted domains [Sofronie, Ihlemann 2007]

Extensions of $\mathcal{EL}$

Extensions of $\mathcal{EL}$ with *n*-ary roles and numeric domains

Example 1: Concepts + maps associating numbers to concepts

Models:

Set-theoretic: $(\mathcal{P}(D), \mathbb{R}, \{\cap, \text{MaxCost}\} \cup \{f_r\}_{r \in N_R})$

$$\text{MaxCost} : \mathcal{P}(D) \rightarrow \mathbb{R}, \text{ monotone.}$$

Algebraic: $(S, \mathbb{R}, \{\wedge, \text{MaxCost}\} \cup \{f_r\}_{r \in N_R})$ where $(S, \wedge) \in \text{SLat}$, f_r mon.

$$\text{MaxCost} : S \rightarrow \mathbb{R} \quad \text{monotone}$$

Extensions of $\mathcal{EL}$

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Example 2: Usual concepts + numerical concepts

Models: $(S, \mathcal{P}(\mathbb{R}), \wedge, \{f_r\}_{f \in N_R})$ where $(S, \wedge) \in$ SLat, f_r mon.

In general: Locality $\mapsto$ parameterized complexity.

PTIME results e.g. when numerical concepts are interpreted over a convex fragment of interval arithmetic (& TBox in similar fragment)

Example: $(S, \text{Int}(\mathbb{R}), \wedge, \{f_r\}_{f \in N_R})$ where $(S, \wedge) \in$ SLat, f_r mon.

Int($\mathbb{R}$) in the Ord-Horn fragment of Allen's interval arithmetic (translate to Horn clauses over atoms $x \leq y$, $x = y$).

TBoxes: Ord-Horn clauses

Extensions of $\mathcal{EL}$

Example 2: Sorts: con, num; roles price, has-weight-price

Interpretation $\mathcal{I} = (D, \mathbb{R}, .^{\mathcal{I}})$

C	$\mapsto$	$C^{\mathcal{I}} \subseteq D$
N	$\mapsto$	$N^{\mathcal{I}} \subseteq \text{Int}(\mathbb{R})$
		$\uparrow n^{\mathcal{I}} = \{x \in \mathbb{R} \mid x \geq n\}$
		$\downarrow n^{\mathcal{I}} = \{x \in \mathbb{R} \mid x \leq n\}$
		$[n, m]^{\mathcal{I}} = \{x \in \mathbb{R} \mid n \leq x \leq m\}$
price	$\mapsto$	$\text{price}^{\mathcal{I}} \subseteq D \times \mathbb{R}$
has-weight-price	$\mapsto$	$\text{has-weight-price}^{\mathcal{I}} \subseteq D \times \mathbb{R} \times \mathbb{R}$

Complex concepts

$\exists \text{price}.\!\uparrow\! n = \{x \mid \exists k \geq n : \text{price}(x, k)\}$

the class of all individuals with some price greater than n.

$\exists$ has-weight-price.$([y, y], \downarrow\text{p}) = \{x \mid \exists y'{=}\text{y}, \exists p'{\leq}\text{p and has-weight-price}(x, y', p')\}$

all individuals with weight y for which a price below p exist.

Extensions of $\mathcal{EL}$

Concepts	car, truck, affordable, C, $\downarrow n_1$, $\uparrow m_1$, $\downarrow n$, $\uparrow m$
Roles	price, has-weight-price, weight
TBox	$\exists \text{price}(\downarrow n_1) \sqsubseteq \text{affordable}$ $\exists \text{weight}(\uparrow m_1) \sqcap \text{car} \sqsubseteq \text{truck}$ $\downarrow n_1 \sqsubseteq \downarrow n \quad \uparrow m_1 \sqsubseteq \uparrow m$ $C \sqsubseteq \text{car}$ $\text{has-weight-price}(\uparrow m, \downarrow n) \sqsubseteq \exists \text{price}(\downarrow n) \sqcap \exists \text{weight}(\uparrow m)$ $C \sqsubseteq \exists \text{has-weight-price}(\uparrow m, \downarrow n)$
Query	$C \sqsubseteq \text{affordable} \sqcap \text{truck}$

Extensions of $\mathcal{EL}$

TBox	$\exists \text{price}(\downarrow n_1) \sqsubseteq \text{affordable}$ $\exists \text{weight}(\uparrow m_1) \sqcap \text{car} \sqsubseteq \text{truck}$ $\downarrow n_1 \sqsubseteq \downarrow n \quad \uparrow m_1 \sqsubseteq \uparrow m$ $C \sqsubseteq \text{car}$ $\text{has-weight-price}(\uparrow m, \downarrow n) \sqsubseteq \exists \text{price}(\downarrow n) \sqcap \exists \text{weight}(\uparrow m)$ $C \sqsubseteq \exists \text{has-weight-price}(\uparrow m, \downarrow n)\}$
Query	$C \sqsubseteq \text{affordable} \sqcap \text{truck}$

Purification; Use locality

Def	C_{num}	C_{concept}	Mon
$f_{\text{price}}(\downarrow n_1) = c_1$	$\downarrow n \leq \downarrow n_1$	$d_1 \leq \text{affordable}$	$\downarrow n_1 \leq \downarrow n \rightarrow c_1 \leq c$
$f_{\text{price}}(\downarrow n) = c$	$\uparrow m \leq \uparrow m_1$	$d_1 \wedge \text{car} \leq \text{truck}$	$\downarrow n_1 \geq \downarrow n \rightarrow c_1 \geq c$
$f_{\text{weight}}(\uparrow m_1) = d_1$		$e \leq c \wedge d$	$\uparrow m_1 \leq \uparrow m \leq d_1 \leq d$
$f_{\text{weight}}(\uparrow m) = d$		$C \leq \text{car}$	$\uparrow m_1 \geq \uparrow m \leq d_1 \geq d$
$f_{\text{h-w-p}}(\uparrow m, \downarrow n) = e$		$C \leq e$	
		$C \not\leq \text{affordable} \wedge \text{truck}$	

Extensions of $\mathcal{EL}^+$

The results established for $\mathcal{EL}^+$ extend to *n*-ary roles

Role inclusions can be again modeled by axioms at the algebraic level.

$$r_1 \circ r_2 \sqsubseteq r \quad \mapsto \quad \begin{array}{l} \text{If } (x_1, \dots, x_n) \in r_1^{\mathcal{I}} \quad \text{and} \\ \quad (x_n, \dots, x_{n+k}) \in r_2^{\mathcal{I}} \\ \text{then } (x_1, \dots, x_{n-1}, x_{n+1}, \dots, x_{n+k}) \in r^{\mathcal{I}}. \end{array}$$

$$f_{r_1}(x_2, \dots, x_{n-1}, f_{r_2}(x_{n+1}, \dots, x_{n+k})) \subseteq f_r(x_2, \dots, x_{n-1}, x_{n+1}, \dots, x_{n+k}).$$

Stable locality results can also be proved in this case.

Conclusions

- Alternative proof of tractability of $\mathcal{EL}, \mathcal{EL}^+$ based on a notion of locality

  $\mapsto$ a hierarchical reduction to SAT in the case of $\mathcal{EL}$

  $\mapsto$ reduction to SAT of ground Horn formulae in the case of $\mathcal{EL}^+$

- Identify tractable extensions of $\mathcal{EL}$ and $\mathcal{EL}^+$

  $\mapsto$ with n-ary roles and/or numerical domains.

Conclusions

- Alternative proof of tractability of $\mathcal{EL}, \mathcal{EL}^+$ based on a notion of locality

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- Identify tractable extensions of $\mathcal{EL}$ and $\mathcal{EL}^+$

  $\mapsto$ with *n*-ary roles and/or numerical domains.

Future work:

- Which axioms are preserved when embedding partial into total models
  Links with canonical extensions?
- Test our implementation on large $\mathcal{EL}$ ontologies for comparisons
- Use results on interpolation/interpolant generation in local extensions
  [VS IJCAR'06] for studying modularity issues in distributed ontologies