On the intermediate logic of open subsets of metric spaces

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Advances In Modal Logic, 2008

Introduction

In this paper we introduce and study a new intermediate logic $ML_{O(\mathcal{X})}$, which can be defined

- as a logic of a Kripke frame, where elements are open subsets of a metric space.
- as a logic of a Heyting algebra, where elements are *information types* (this concept will be explained later).

Kripke frames

Definition

An (intuitionistic) *Kripke frame* is a partially ordered set $(F, \leq)$.

Definition

A *Kripke model* is a Kripke frame with a valuation θ (a function which maps propositional variables to upward-closed subsets of the Kripke frame).

Kripke semantics

The notion of a propositional formula being true at some point $w \in F$ of a model $M = (F, \theta)$ is defined recursively as follows:

$$\text{For propositional letter } p_i,\ M, w \vDash p_i \text{ iff } w \in \theta(p_i)$$
$$M, w \vDash \psi \wedge \chi \text{ iff } M, w \vDash \psi \text{ and } M, w \vDash \chi$$
$$M, w \vDash \psi \vee \chi \text{ iff } M, w \vDash \psi \text{ or } M, w \vDash \chi$$
$$M, w \vDash \psi \rightarrow \chi \text{ iff for any } w' \geq w,\ M, w \vDash \psi \text{ implies } M, w' \vDash \chi$$
$$M, w \nvDash \bot$$

- A formula *is true* in a model M if it is true in each point of M.
- A formula is valid in a frame F if it is true in all models based on that frame.

Basic definitions

Definition

Let $u \in F$ be a point of a Kripke frame $(F, \leq)$. Then the Kripke frame $F^u = \{v \in F | u \leq v\}$ with the ordering $\leq$ is called a *cone*.

Definition

The surjective mapping h from frame F to frame G is called a *p-morphism* iff $\forall u \in F\ h(F^u) = G^{h(u)}$.
If there is a p-morphism from F to G, it is denoted by $F \twoheadrightarrow G$.

Lemma

1. $L(F) \subseteq L(F^u)$.
2. *If* $F \twoheadrightarrow G$ *then* $L(F) \subseteq L(G)$.

Intermediate logics

- The set of formulas which are valid on every Kripke frame is *intuitionistic logic* $\mathbf{Int}$.
- An *intermediate logic* is an extension of $\mathbf{Int}$ closed under modus ponens and substitution. Every consistent intermediate logic is contained in *classical logic* $\mathbf{CL}$.
- The set $L(F)$ of all formulas valid in a given frame F forms an intermediate logic.

Logics ML and ML_1

For example, the logics ML and ML_1, which will be mentioned often in this paper are defined as follows:

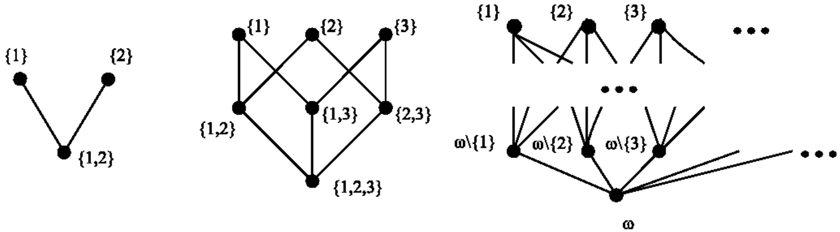
Definition

Let X be a set, then $P_1(X)$ is the Kripke frame $(2^X \setminus \emptyset, \supseteq)$ (non-empty subsets of X ordered by converse inclusion).

- *Medvedev's logic of finite problems* ML is $\bigcap\{L(P_1(X)) | X \text{ is finite}\}$
- The *logic of infinite problems* ML_1 is $L(P_1(\omega))$ (or $L(P_1(X))$ for any infinite X).

$ML_1 \subseteq ML$, but it is an open problem whether $ML_1 = ML$ or not.

Frames $P_1(X)$

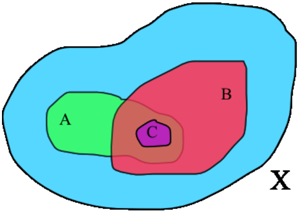


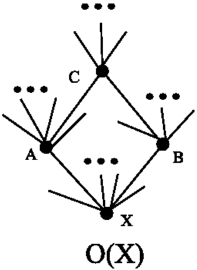
$P_1(\{1,2\})$ $P_1(\{1,2,3\})$ $P_1(\omega)$

Definition of $ML_{O(\mathcal{X})}$ in Kripke semantics

Definition

- Let $\mathcal{X}$ be a topological space, and $O(\mathcal{X})$ a set of non-empty open sets in $\mathcal{X}$.
- Then $(O(\mathcal{X}), \supseteq)$ is a Kripke frame with the least element $\mathcal{X}$.
- Logic $ML_{O(\mathcal{X})}$ is defined as the logic of this frame.





Information types

- *Information types* were introduced by Yuri Medvedev in 1979.
- According to his approach, *information* is understood as a property of elements of some base set Ω (a subset of Ω).
- *Information type* is an arbitrary set of informations. For example, "residue class modulo m" is an information type on ω.
- We say that type σ_1 is *transformable* to σ_2 ($\sigma_1 \geq \sigma_2$) iff $\forall E_1 \in \sigma_1 \exists E_2 \in \sigma_2 (E_1 \subseteq E_2)$. For example, the type of residue classes modulo 9 is "more informative" than classes modulo 3.

- A type σ is *regular* iff $\forall E_1 \forall E_2\ E_1 \in \sigma\ \&\ E_2 \subseteq E_1 \Rightarrow E_2 \in \sigma$.
- Regular types ordered by $\geq$ form a Heyting algebra $I(\Omega)$, which is isomorphic to the Heyting algebra of upward-closed subsets of $P_1(\Omega)$.
- Therefore, the logic of $I(\Omega)$ is the logic of $P_1(\Omega)$.
- So, the logics ML and ML_1 can also be defined in terms of structures related to information types.

"Imprecise" information types

Now, let us modify the definition of information types a bit.

- If $\mathcal{X}$ is a topological space, it can be natural to understand information about elements of $\mathcal{X}$ as an open set in $\mathcal{X}$.
- For example, any physical measurement always has a margin of error, so any information about the location of physical objects is always an open set.
- Now, if we consider regular information types that contain only this kind of information, they will also form a Heyting algebra, and it is isomorphic to the Heyting algebra of upward-closed subsets of $O(\mathcal{X})$.
- Thus, the logic of this Heyting algebra is $ML_{O(\mathcal{X})}$.

Theorem 1

Theorem

Let $\mathcal{X}$ be an infinite metric space. Then $ML_{O(\mathcal{X})} \subseteq ML_1$.
If $\mathcal{X}$ consists only of isolated points and limits of isolated points then $ML_{O(\mathcal{X})} = ML_1$.

Notes on the proof of Theorem 1

- We prove this by constructing a p-morphism from $O(\mathcal{X})$ to $P_1(\omega)$, which leads to $ML_{O(\mathcal{X})} \subseteq ML_1$.
- To construct it, we divide $\mathcal{X}$ into 3 disjoint parts:
  - $\mathcal{X}_1$ — isolated points.
  - $\mathcal{X}_2$ — the limits of isolated points.
  - $\mathcal{X}_3$ — limit points that are not in $\mathcal{X}_2$.
- It is also possible to construct a p-morphism from $P_1(Y)$ to $O(\mathcal{X})$ iff $\mathcal{X}_3$ is empty, which leads to $ML_{O(\mathcal{X})} = ML_1$ for such $\mathcal{X}$.

Theorem 2

Definition

A *k-formula* is a formula that does not contain propositional letters besides $p_1,\dots,p_k$.

Theorem

Let $\mathcal{X}$ be an infinite metric space. Then $ML_{O(\mathcal{X})}$ is not axiomatizable in finite number of variables, that is for any $k \in \mathbb{N}$ $ML_{O(\mathcal{X})}$ is not axiomatizable by any set of k-formulas.

Corollary

$ML_{O(\mathcal{X})}$ is not finitely axiomatizable.

Theorem 3

Theorem

Let $\mathcal{X}$ be an infinite metric space and F a finite frame with the least element, and no $u \in F$ has exactly one immediate successor. Then

$$X(F) \in ML_{O(\mathcal{X})} \iff X(F) \in ML$$

$X(F)$ is a propositional formula, called *Jankov (characteristic) formula*, defined for any finite frame with the least element.

Corollary

Let L be an intermediate logic, and $ML_{O(\mathcal{X})} \subseteq L \subseteq ML$ for some infinite metric space $\mathcal{X}$. Then L is not axiomatizable in finite number of variables.

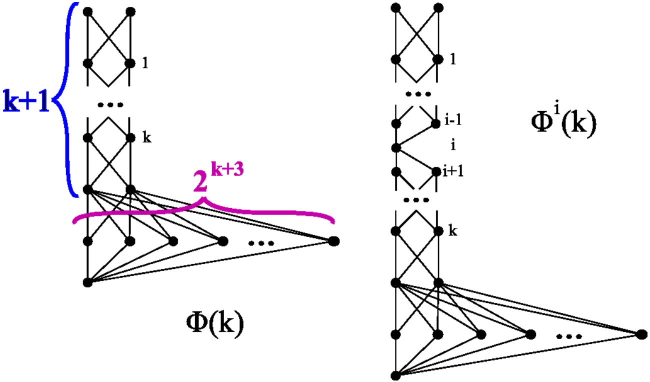
Overview of the proof

We use the method that was used to prove that ML is not axiomatizable in finite number of variables (Maksimova, Skvortsov, Shehtman, 1979), which is also applicable to ML_1 and other similar logics.
We find two families of frames $\Phi(k)$ and $\Phi^i(k)$ such that

- $ML_{O(\mathcal{X})}$ is valid on $\Phi^i(k)$ for each i, but not on $\Phi(k)$.
- When a k-formula is valid on all $\Phi^i(k)$, then it is valid on $\Phi(k)$.

It follows easily that $ML_{O(\mathcal{X})}$ is not axiomatizable in k-formulas.

Frames $\Phi(k)$ and $\Phi^i(k)$



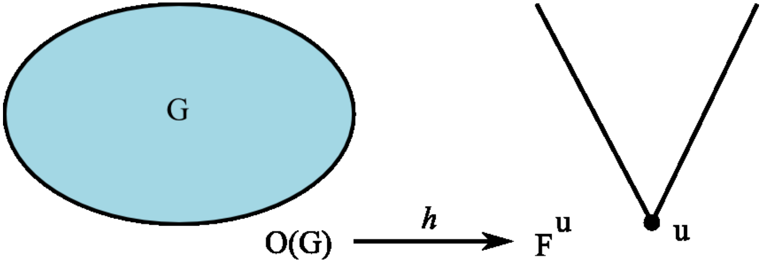
- $\Phi^i(k)$ are easily shown to be ML-frames, and thus $ML_{O(\mathcal{X})}$-frames.
- As for $\Phi(k)$ we show that there is no p-morphism to it from $O(\mathcal{X})$.

Let F be a finite frame with the least element. For $u \in F$:

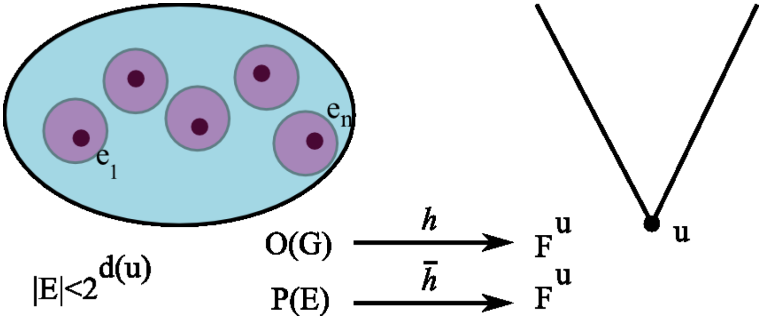
- $br(u)$ — the set of its immediate successors.
- $d(u)$ — the maximum length of an increasing chain starting at u.

The bulk of the proof consists of showing that if F is such that $\forall u \in F |br(u)| \neq 1$, any p-morphism from $O(\mathcal{X})$ to F, can be reduced to a p-morphism from $P_1(E)$ to F, where E is some finite set.
Unless $\forall u \in F |br(u)| < 2^{d(u)}$ such a p-morphism cannot exist, and $\Phi(k)$ doesn't fit this requirement.

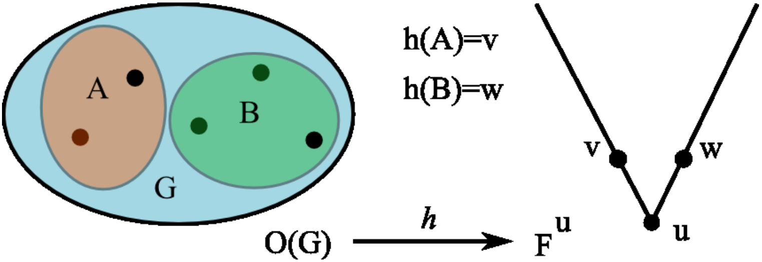
Proof of Theorems 2,3



Proof of Theorems 2,3



Proof of Theorems 2,3



Conclusions

To summarise the results, we have discovered that:

- Logics $ML_{O(\mathcal{X})}$ (for different infinite metric spaces $\mathcal{X}$) are subsets of ML_1 and ML.
- $ML_{O(\mathcal{X})}$ are not finitely axiomatizable
- We cannot distinguish them from ML or each other by using formulas of the form $X(F)$, where F is a finite frame with a particular property.

Open Questions

Some of the open questions are:

1. Is (for example) $ML_{O(\mathbb{R})}$ different from ML_1 and ML?
2. Is $ML_{O(\mathbb{R})}$ recursively axiomatizable?
3. Is $ML_{O(\mathbb{R})}$ decidable?
4. What is the intersection of the logics $ML_{O(\mathcal{X})}$? Does there exist a metric space $\mathcal{X}_0$ such that $ML_{O(\mathcal{X}_0)} \subseteq ML_{O(\mathcal{X})}$ for all metric spaces $\mathcal{X}$?