PSPACE-decidability of Japaridze’s Polymodal Logic

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Satisfiability on ordinal sums of transitive frames

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Japaridze’s Polymodal Logic

GLP is the normal propositional modal logic with countably many modalities:

\[ \Box_i (\Box_i p \rightarrow p) \rightarrow \Box_i p \text{ for all } i \]
\[ \Box_i p \rightarrow \Box_j p \text{ for all } i < j \]
\[ \Diamond_i p \rightarrow \Box_j \Diamond_j p \text{ for all } i < j \]

[Japaridze, Ignatiev, Boolos, Beklemishev,...]
Japaridze’s Polymodal Logic
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Main idea

If a class of frames can be represented as a class of ordinal sums of ”simple” frames, then it is also ”simple”.
Decision procedures for ordinal sums of frames
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1. Monomodal case
Decision procedures for ordinal sums of frames

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   - Ordinal sums of transitive frames
Decision procedures for ordinal sums of frames

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   - Ordinal sums of transitive frames
   - Truth-preserving transformations for ordinal sums of frames
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   - Truth-preserving transformations for ordinal sums of frames
   - Conditional satisfiability and moderate classes of frames
Decision procedures for ordinal sums of frames

1. Monomodal case
   ▶ Ordinal sums of transitive frames
   ▶ Truth-preserving transformations for ordinal sums of frames
   ▶ Conditional satisfiability and moderate classes of frames

2. Polymodal case
Ordinal sums of frames
Ordinal sums of frames

$I = (W, R)$ is a finite partial order, $W = \{w_1, \ldots, w_n\}$

$F_1 = (V_1, S_1), \ldots, F_n = (V_n, S_n)$ are transitive frames
Ordinal sums of frames

\( I = (W, R) \) is a finite partial order, \( W = \{w_1, \ldots, w_n\} \)
\( F_1 = (V_1, S_1), \ldots, F_n = (V_n, S_n) \) are transitive frames

\[
I = (W, R) \quad \text{and} \quad I[(F_1, \ldots, F_m)/(w_1, \ldots, w_m)]
\]

\[
I[(F_1, \ldots, F_m)/(w_1, \ldots, w_m)] = (\overline{W}, \overline{R}):
\]
\[
\overline{W} = (\{w_1\} \times V_1) \cup \cdots \cup (\{w_m\} \times V_m)
\]
\[
(w', v') \overline{R} (w'', v'') \iff (w' \neq w'' \& w'Rw'') \quad \text{or} \quad (w' = w'' = w_i \& v'S_iv'')
\]
Ordinal sums of frames

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$F_1 = (V_1, S_1), \ldots, F_n = (V_n, S_n)$ are transitive frames

For a class $\mathcal{F}$ of frames,

$I[\mathcal{F}] = \{I[(F_1, \ldots, F_m)/(w_1, \ldots, w_m)] \mid F_1, \ldots, F_m \in \mathcal{F}\}$
Ordinal sums of frames

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For a class $\mathcal{I}$ of finite partial orders,
$\mathcal{I}[\mathcal{F}] = \bigcup\{I[\mathcal{F}] \mid I \in \mathcal{I}\}$
Ordinal sums of frames

Example: skeleton
Every transitive frame can be considered as an ordinal sum of its clusters.
Ordinal sums of frames

Example: transitive frames

\( \mathcal{PO} \) denotes the class of all finite (strict or non-strict) partial orders.

For \( n \geq 1 \), \( C_n = (W_n, W_n \times W_n) \), where \( W_n = \{1, \ldots, n\} \);
\( C_0 \) denotes the irreflexive singleton \((\{0\}, \emptyset)\).

Let \( \mathcal{F} = \{C_0, C_1, C_2 \ldots\} \), \( \mathcal{G} = \{C_1, C_2, \ldots\} \).
Then (up to isomorphisms):
\( \mathcal{PO}[\mathcal{F}] \) is the class of all finite \( K_4 \)-frames,
\( \mathcal{PO}[\mathcal{G}] \) is the class of all finite \( S_4 \)-frames.
Truth-preserving transformations for ordinal sums of frames

Treelike frames

\( \mathcal{T} \) denotes the class of all finite transitive trees.

**Lemma**

Let \( \mathcal{F} \) be a class of frames, \( I \) be a finite rooted partial order. Then for any \( H \in I[\mathcal{F}] \) there exists a tree \( T \in \mathcal{T} \) such that for some \( H' \in \mathcal{T}[\mathcal{F}] \) we have \( H' \rightarrow H \).

**Corollary**

\( \varphi \) is \( PO[\mathcal{F}] \)-satisfiable \( \Rightarrow \) \( \varphi \) is \( \mathcal{T}[\mathcal{F}] \)-satisfiable
Truth-preserving transformations for ordinal sums of frames

Restricting height and branching

\[ \langle \varphi \rangle \] denotes the cardinality of \( Sub(\varphi) \).

**Well-known fact:** Any \( K_4 \)-satisfiable formula \( \varphi \) is satisfiable in some finite frame with the height and the branching of its skeleton not more then \( \langle \varphi \rangle \).
Truth-preserving transformations for ordinal sums of frames

Restricting height and branching

Let $\mathcal{T}_{n,b}$ denotes the class of transitive trees with the height not more then $h$ and the branching not more then $b$:

$$\mathcal{T}_{h,b} = \{ T \in \mathcal{T} \mid Ht(T) \leq h, \ Br(T) \leq b \}.$$ 

Lemma

Let $\mathcal{F}$ be a class of transitive frames. If a formula $\varphi$ is $\mathcal{PO}[\mathcal{F}]$-satisfiable, then $\varphi$ is $\mathcal{T}_{\langle \varphi \rangle, \langle \varphi \rangle}[\mathcal{F}]$-satisfiable.
Selective filtration

A model \(((W', R'), \theta')\) is a **weak submodel** of \(((W, R), \theta)\), if \(W' \subseteq W\), \(R' \subseteq R\), \(\theta(p) = \theta'(p) \cap W'\) for any propositional variable \(p\).

**Definition**

Let \(M\) be a model, \(\Psi\) a set of formulas closed under subformulas. A weak submodel \(M'\) of \(M\) is called a **selective filtration of \(M\) through \(\Psi\)**, if for any \(w \in M'\), for any formula \(\psi\), we have

\[
\Diamond \psi \in \Psi \& M, w \models \Diamond \psi \Rightarrow \exists u \in R'(x) \ M, u \models \psi,
\]

where \(R'\) is the accessibility relation of \(M'\).

**Lemma**

*If \(M'\) is a selective filtration of \(M\) through \(\Psi\), then for any \(w \in M'\), for any \(\psi \in \Psi\), we have*

\[
M, w \models \psi \iff M', w \models \psi.
\]
Conditional satisfiability

M is a Kripke model

\[
\begin{align*}
M, w \not\models \bot; \\
M, w \models p & \iff w \in \theta(p); \\
M, w \models \varphi \rightarrow \psi & \iff M, w \not\models \varphi \text{ or } M, w \models \psi; \\
M, w \models \Diamond \varphi & \iff \exists v (wRv \& M, v \models \varphi). \\
\end{align*}
\]

"\( \varphi \) is true at \( w \) in \( M \)."
Conditional satisfiability

M is a Kripke model, $\Psi$ is a set of formulas

\[
\begin{align*}
(M, w|\Psi) \not\models \bot & \iff w \in \theta(p); \\
(M, w|\Psi) \models p & \iff (M, w|\Psi) \not\models \varphi \text{ or } (M, w|\Psi) \models \psi \\
(M, w|\Psi) \models \varphi \rightarrow \psi & \iff \exists v (wRv \& M, v \models \varphi) \\
(M, w|\Psi) \models \Diamond \varphi & \iff \varphi \in \Psi \text{ or } \Diamond \varphi \in \Psi
\end{align*}
\]

“$\varphi$ is true at $w$ in $M$ under the condition $\Psi$ ”.
Consider transitive models $M_0$, $M$, their ordinal sum $M_0 + M$, and a formula $\varphi$. Put

$$\Psi = \{ \psi \in \text{Sub}(\varphi) \mid M, \nu \models \psi \text{ for some } \nu \}$$

Then for any $w \in M_0$,

$$M_0 + M, w \models \varphi \iff (M_0, w|\Psi) \models \varphi$$

More generally: for any set of formulas $\Phi$,

$$(M_0 + M, w|\Phi) \models \varphi \iff (M_0, w|\Psi \cup \Phi) \models \varphi$$

$$(M_0 + M, w|\Phi) \models \varphi \quad (M_0, w|\Psi \cup \Phi) \models \varphi$$
Moderate classes of frames

For a cone $F$, $F \models \varphi$ means that $\varphi$ is satisfiable at a root of $F$;
$F \models \psi \models \varphi$ means $(M, w|\psi) \models \varphi$ for some model $M$ based on $F$, where $w$ is a root of $F$.
For a class of cones $\mathcal{F}$, $\mathcal{F} \models \psi \models \varphi$ means that $F \mid \psi \models \varphi$ for some $F \in \mathcal{F}$

Definition
A sequence $(\mathcal{F}_n)_{n \in \mathbb{N}}$ of sets of rooted frames is called $d$-moderate for $d \in \mathbb{N}$, if there exists an algorithm such that for any formula $\varphi$ and any $\psi, \Phi \subseteq \text{Sub}(\varphi)$ it decides whether

$$\mathcal{F}_{\langle \varphi \rangle} \mid \psi \models \bigwedge \Phi$$

in space $O(\langle \varphi \rangle^d)$. 
Moderate classes of frames: examples

Definition
A sequence \((\mathcal{F}_n)_{n \in \mathbb{N}}\) of sets of rooted frames is called \textit{d-moderate} for \(d \in \mathbb{N}\), if there exists an algorithm such that for any formula \(\varphi\) and any \(\psi, \Phi \subseteq \text{Sub}(\varphi)\) it decides whether

\[
\mathcal{F}_{\langle \varphi \rangle} \mid \psi \vdash \bigwedge \Phi
\]

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Moderate classes of frames: examples

Definition
A sequence \((\mathcal{F}_n)_{n \in \mathbb{N}}\) of sets of rooted frames is called *d-moderate* for \(d \in \mathbb{N}\), if there exists an algorithm such that for any formula \(\varphi\) and any \(\psi, \phi \subseteq \text{Sub}(\varphi)\) it decides whether

\[
\mathcal{F}_{\langle \varphi \rangle} \models \psi \models \bigwedge \phi
\]

in space \(O(\langle \varphi \rangle^d)\).

Example: \((\mathcal{F}_n)_{n \in \mathbb{N}}\) is moderate, if:

- \(\mathcal{F}_n\) is the set of all (non-degenerate) clusters with cardinality not more than \(n\):
  
  for all \(n\) \(\mathcal{F}_n = \{C_0, \ldots, C_n\}\) or for all \(n\) \(\mathcal{F}_n = \{C_1, \ldots, C_n\}\);

- \(\mathcal{F}_n\) consists of a single frame which is a singleton:
  
  for all \(n\) \(\mathcal{F}_n = \{C_0\}\) or for all \(n\) \(\mathcal{F}_n = \{C_1\}\).
Moderate classes of frames: examples

Definition
A sequence \((\mathcal{F}_n)_{n \in \mathbb{N}}\) of sets of rooted frames is called \(d\)-moderate for \(d \in \mathbb{N}\), if there exists an algorithm such that for any formula \(\varphi\) and any \(\Psi, \Phi \subseteq \text{Sub}(\varphi)\) it decides whether

\[
\mathcal{F}_{\langle \varphi \rangle} \models \Psi \supseteq \bigwedge \Phi
\]

in space \(O(\langle \varphi \rangle^d)\).

For classes \(\mathcal{F}, \mathcal{G}\), put

\[
\mathcal{F} + \mathcal{G} = \{ F + G \mid F \in \mathcal{F}, G \in \mathcal{G} \}
\]

Lemma
If \((\mathcal{F}_n)_{n \in \mathbb{N}}, (\mathcal{G}_n)_{n \in \mathbb{N}}\) are moderate, then \((\mathcal{F}_n + \mathcal{G}_n)_{n \in \mathbb{N}}\) are moderate.
Main lemma

If \((\mathcal{F}_n)_{n \in \mathbb{N}}\) is \(d\)-moderate sequence of sets of cones, and \(P\) is a polynomial of degree \(d'\), then the sequence \((\mathcal{T}_{P(n)}, P(n)[\mathcal{F}_n])_{n \in \mathbb{N}}\) is \(\max\{2 + d', d\}\)-moderate.
Algorithm

Let $\text{SatSimple}(\varphi, \Phi, \Psi)$ decide whether $\mathcal{F} | \psi \vdash \bigwedge \Phi$ for any $\varphi$, $\Phi, \Psi \subseteq \text{Sub}(\varphi)$ in space $f(\langle \varphi \rangle)$.

Then the following algorithm decides whether $\mathcal{T}_{h,b}[\mathcal{F}] | \psi \vdash \bigwedge \Phi$ for any $\varphi$, $\Phi, \Psi \subseteq \text{Sub}(\varphi)$, $h, b > 0$ in space $O(f(\langle \varphi \rangle) + \langle \varphi \rangle bh)$:

Function $\text{SatTree}(\varphi; \Phi, \Psi; h, b)$ returns boolean;
Begin
  if $\text{SatSimple}(\varphi, \Phi, \Psi)$ then return(true);
  if $h > 1$ then
    for every integer $b'$ such that $1 \leq b' \leq b$
      for every $\Phi_1, \ldots, \Phi_{b'} \subseteq \text{Sub}(\varphi)$
        if $\bigwedge_{1 \leq j \leq b'} \text{SatTree}(\varphi, \psi_j, \Psi, h - 1, b)$ then
          if $\text{SatSimple}(\varphi, \Phi, \Psi \cup \psi_1 \cdots \cup \psi_{b'})$ then
            return(true);
      return(false);
  return(false);
End.
Algorithm

Lemma
Let $F$ be a class of frames, and let $G \in \mathcal{I}_{h+1,b}[F]$ for some $h, b \geq 1$. Then $G$ is either isomorphic to a frame $F \in F$ or isomorphic to a frame $F + (G_1 \sqcup \cdots \sqcup G_{b'})$, where $1 \leq b' \leq b$, $F \in F$, $G_1, \ldots, G_{b'} \in \mathcal{I}_{h,b}[F]$. 
Semantic condition

**Theorem**

Suppose that a logic $L$ is characterized by $PO[\mathcal{F}]$ for some class $\mathcal{F}$. If there exists a moderate sequence $(\mathcal{F}_n)_{n \in \mathbb{N}}$ such that $\mathcal{F}_n \subseteq \mathcal{F}$ for all $n \in \mathbb{N}$, and any $L$-satisfiable formula $\varphi$ is $PO[\mathcal{F}(\varphi)]$-satisfiable, then $L$ is in PSPACE.

**Corollary**

If $L = L(PO[\mathcal{F}])$ for some finite class of finite cones $\mathcal{F}$, then $L$ is in PSPACE.
Examples
Consider the logics $\mathbf{K}4$, $\mathbf{S}4$, Gödel-Löb logic $\mathbf{GL}$, and Grzegorczyk logic $\mathbf{GRZ}$. They are well-known to be PSPACE-decidable. Let us illustrate, how this fact follows from the above theorem.
Examples
Consider the logics $K4$, $S4$, Gödel-Löb logic $GL$, and Grzegorczyk logic $GRZ$. They are well-known to be PSPACE-decidable. Let us illustrate, how this fact follows from the above theorem.

$GRZ$ ($GL$) is the logic of all finite non-strict (strict) partial orders:
$GRZ = L(PO[\{C_1\}]), \quad GL = L(PO[\{C_0\}]).$

By the above corollary, $GRZ$ and $GL$ are in PSPACE.
Examples
Consider the logics $K4$, $S4$, Gödel-Löb logic $GL$, and Grzegorczyk logic $GRZ$. They are well-known to be PSPACE-decidable. Let us illustrate, how this fact follows from the above theorem.

Any $K4$-satisfiable formula is satisfiable at some finite transitive frame $F$ such that the cardinality of any cluster in $F$ does not exceed $\langle \varphi \rangle$.

Put
\[
F_{n}^{K4} = \{C_{0}, \ldots, C_{n}\}, \quad F_{n}^{S4} = \{C_{1}, \ldots, C_{n}\}.
\]

Then for any $\varphi$ we have:

$\varphi$ is $K4$-satisfiable iff $\varphi$ is $PO[F_{\langle \varphi \rangle}^{K4}]$-satisfiable,

$\varphi$ is $S4$-satisfiable iff $\varphi$ is $PO[F_{\langle \varphi \rangle}^{S4}]$-satisfiable.

Since the sequences $(F_{n}^{K4})_{n\in\mathbb{N}}$ and $(F_{n}^{S4})_{n\in\mathbb{N}}$ are moderate, then $K4$ and $S4$ are in PSPACE.
Example: logic $LM$

\[
LM = K4 + \Box \top + \Box p_1 \land \Box p_2 \rightarrow \Box (\Box p_1 \land \Box p_2) \text{ (the logic of interval inclusion, compact inclusion, chronological future)}
\]
Example: logic $\mathcal{LM}$

For a transitive finite frame $F$,

$F \models \mathcal{LM} \iff$ every its degenerate cluster has a unique successor $C$, and $C$ is non-degenerate.
Example: logic $\text{LM}$

For a transitive finite frame $F$, $F \models \text{LM} \iff$ every its degenerate cluster has a unique successor $C$, and $C$ is non-degenerate.

For $\mathcal{F} = \{C_0 + C \mid C$ is a finite cluster$\}$, $\mathcal{PO}[\mathcal{F}]$ is the class of all finite $\text{LM}$-frames (up to isomorphisms).
Example: logic LM

For a transitive finite frame $F$, $F \vdash LM \iff$ every its degenerate cluster has a unique successor $C$, and $C$ is non-degenerate.

For $F = \{C_0 + C \mid C \text{ is a finite cluster}\}$, $\mathcal{PO}[F]$ is the class of all finite LM-frames (up to isomorphisms).

LM has the FMP, thus: $\varphi$ is LM-satisfiable $\Rightarrow$ $\varphi$ is $\mathcal{PO}[F_{\langle \varphi \rangle}]$-satisfiable, where $F_n = \{C_0 + C_1, \ldots, C_0 + C_n\}$
Example: logic LM

For a transitive finite frame \( F \),
\( F \models \text{LM} \iff \) every its degenerate cluster has a unique successor \( C \), and \( C \) is non-degenerate.

\[
\begin{array}{c}
\circ \\
/ \\
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/ \\
\circ \\
\end{array}
\]

For \( F = \{ C_0 + C \mid C \text{ is a finite cluster} \} \),
\( \mathcal{PO}[F] \) is the class of all finite LM-frames (up to isomorphisms).

LM has the FMP, thus: \( \varphi \) is LM-satisfiable \( \Rightarrow \) \( \varphi \) is \( \mathcal{PO}[F_{\langle \varphi \rangle}] \)-satisfiable,
where \( F_n = \{ C_0 + C_1, \ldots, C_0 + C_n \} \)

The sequence \( (F_n)_{n \in \mathbb{N}} \) is moderate, thus LM is PSPACE-decidable.
Multimodal case
Ordinal sums of multimodal frames

$I = (W, R)$ is a finite partial order, $W = \{w_1, \ldots, w_n\}$

$F_1 = (W_1, R_1^1, \ldots, R_1^N), \ldots, F_n = (W_n, R_n^1, \ldots, R_n^N)$ are $N$-frames;

$1 \leq k \leq N$. $I[k; (F_1, \ldots, F_m)/(w_1, \ldots, w_m)]$:

For a class $\mathcal{F}$ of $N$-frames, put

$$G[k; \mathcal{F}] = \{I[k; (F_1, \ldots, F_m)/(w_1, \ldots, w_m)] \mid F_1, \ldots, F_n \in \mathcal{F}\},$$

for a class $\mathcal{I}$ of finite partial orders, put

$$\mathcal{I}[k; \mathcal{F}] = \bigcup\{I[k; \mathcal{F}] \mid I \in G\}.$$
Lemma

Let $\mathcal{F}$ be a class of $N$-frames, $1 \leq k \leq N$. If an $N$-formula $\varphi$ is $PO[k; \mathcal{F}]$-satisfiable, then $\varphi$ is $T_{\langle \varphi \rangle, \langle \varphi \rangle}[k; \mathcal{F}]$-satisfiable.
Definition
Let $M$ be an $N$-model. A condition for $M$ is a tuple $\overline{\psi} = (\psi_1, \ldots, \psi_N)$ of sets of $N$–formulas. For an $N$–formula $\varphi$ and a point $w \in M$, we define the truth-relation $(M, w|\overline{\psi}) \models \varphi$ (“$\varphi$ is true at $w$ in $M$ under the condition $\overline{\psi}$”):

$$(M, w|\overline{\psi}) \models \top \quad \Leftrightarrow \quad M, w \models \top$$

$$(M, w|\overline{\psi}) \not\models \bot \quad \Leftrightarrow \quad (M, w|\overline{\psi}) \not\models \varphi \text{ or } (M, w|\overline{\psi}) \models \psi$$

$$(M, w|\overline{\psi}) \models \Diamond_k \varphi \quad \Leftrightarrow \quad \varphi \in \psi_k \text{ or } \Diamond_k \varphi \in \psi_k \text{ or }$$

for some $v \in R_k(w)$ we have $(M, w|\psi_k) \models \varphi$,

where $R_1, \ldots, R_N$ are the accessibility relations in $M$. 
An $N$-frame $G = (W, R_1, \ldots, R_N)$ is called \textit{rooted}, if for some $w \in W$ we have $\{w\} \cup R_1(w) \cup \cdots \cup R_N(w) = W$; $w$ is called a \textit{root} of $G$.

For an $N$-formula $\varphi$, put

$$Sub^*(\varphi) = Sub(\varphi) \cup \{\lozenge_i \psi \mid 1 \leq i, j \leq N, \lozenge_j \psi \in Sub(\varphi)\}.$$ 

\textbf{Definition}

A sequence $(\mathcal{F}_n)_{n \in \mathbb{N}}$ of sets of rooted $N$-frames is called \textit{d-moderate}, if there exists an algorithm such that for any $N$-formula $\varphi$ and any $\Phi, \psi_1, \ldots, \psi_n \subseteq Sub^*(\varphi)$, it decides whether

$$\mathcal{F}_{|Sub^*(\varphi)|} \models (\psi_1, \ldots, \psi_n) \models \bigwedge \Phi$$

in polynomial space.
Theorem

If \((F_n)_{n \in \mathbb{N}}\) is \(d\)-moderate sequence of sets of rooted \(N\)-frames, \(1 \leq k \leq N\), \(P\) is a polynomial of degree \(d'\), then the sequence

\[
(T_{P(n), P(n)}[k; F_n])_{n \in \mathbb{N}}
\]

is \(\max\{2 + d', d\}\)-moderate.
Japaridze’s Polymodal Logic

For an $N$-frame $F = (W, R_1, \ldots, R_N)$ let

$F_+ \text{ denote the } (N+1)\text{-frame } (W, \emptyset, R_1, \ldots, R_N),$

$F_\infty \text{ denote the frame } (W, R_1, \ldots, R_N, \emptyset, \emptyset, \ldots )$

*Hereditary strict orders:*

$\mathcal{F}^{(1)}$ is the class of all finite strict partial orders;

$\mathcal{F}^{(N+1)} = \mathcal{PO}[1; \mathcal{G}^{(N)}], \text{ where } \mathcal{G}^N = \{F_+ \mid F \in \mathcal{F}^{(N)}\}.$

$\mathcal{F}^J = \{F_\infty \mid F \in \mathcal{F}^{(N)} \text{ for some } N\}.$

**Theorem (Beklemishev, 2007)**

There exists a polynomial-time translation $f$ such that for any formula $\varphi$ we have

$$GLP \vdash \varphi \iff \mathcal{F}^J \models f(\varphi).$$
Let $\mathcal{T}^{(1)}_{h,b} = \mathcal{T}_{h,b}[\{C_0\}]$, i.e., $\mathcal{T}^{(1)}_{h,b}$ is the class (up to isomorphisms) of all finite transitive irreflexive trees with the height not more than $h$ and the branching not more than $b$. Put

$$\mathcal{T}^{(N+1)}_{h,b} = \mathcal{T}_{h,b}[1; \{F_+ \mid F \in \mathcal{T}^{(N)}_{h,b}\}].$$

**Corollary**

If an $N$-modal formula $\varphi$ is $\mathcal{F}^J$-satisfiable, then $\varphi$ is $\mathcal{T}^{(N)}_{N,N}$-satisfiable.

**Theorem**

The satisfiability problem for $\mathcal{F}^J$ is in PSPACE.

**Theorem**

Japaridze’s Polymodal Logic is PSPACE-decidable.