PSPACE-decidability of Japaridze's Polymodal Logic

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Satisfiability on ordinal sums of transitive frames

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Japaridze's Polymodal Logic

GLP is the normal propositional modal logic with countably many modalities:

$\Box_i(\Box_i p \rightarrow p) \rightarrow \Box_i p$ for all i
$\Box_i p \rightarrow \Box_j p$ for all $i < j$
$\Diamond_i p \rightarrow \Box_j \Diamond_j p$ for all $i < j$

[Japaridze, Ignatiev, Boolos, Beklemishev,...]

Japaridze's Polymodal Logic

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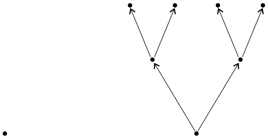

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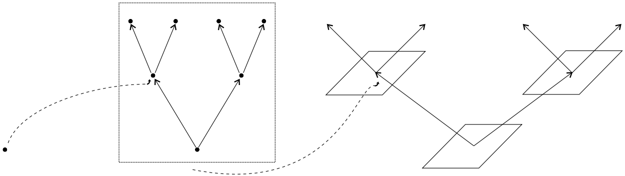

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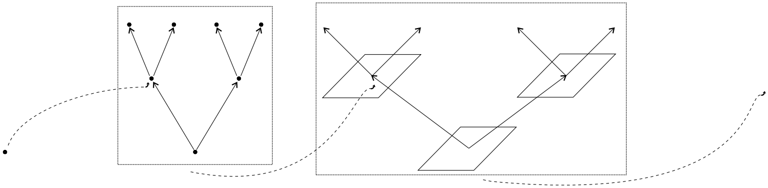

Main idea

If a class of frames can be represented as a class of ordinal sums of "simple" frames, then it is also "simple".

Decision procedures for ordinal sums of frames

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1. Monomodal case

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   - Ordinal sums of transitive frames

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   - Ordinal sums of transitive frames
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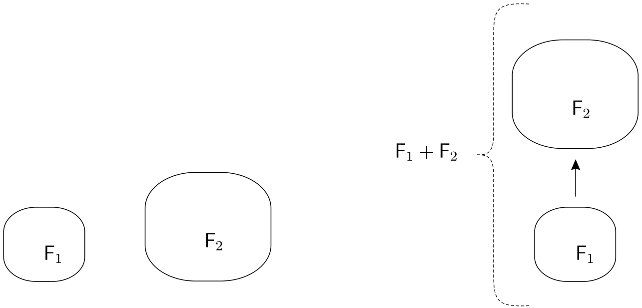
   - Conditional satisfiability and moderate classes of frames
2. Polymodal case

Ordinal sums of frames



Ordinal sums of frames

$\mathsf{I} = (W, R)$ is a finite partial order, $W = \{w_1, \ldots, w_n\}$
$\mathsf{F}_1 = (V_1, S_1), \ldots, \mathsf{F}_n = (V_n, S_n)$ are transitive frames

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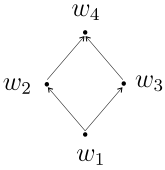

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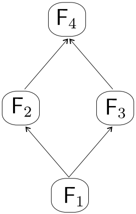

$\mathsf{I}[(\mathsf{F}_1, \dots, \mathsf{F}_m)/(w_1, \dots, w_m)]$

$\mathsf{I}[(\mathsf{F}_1, \dots, \mathsf{F}_m)/(w_1, \dots, w_m)] = (\overline{W}, \overline{R})$:

$\overline{W} = (\{w_1\} \times V_1) \ \cup \ \cdots \cup \ (\{w_m\} \times V_m)$

$(w', v')\overline{R}(w'', v'') \Leftrightarrow (w' \neq w'' \ \& \ w'Rw'') \ \text{ or } \ (w' = w'' = w_i \ \& \ v'S_iv'')$

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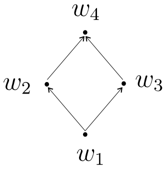

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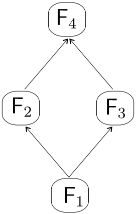

$\mathsf{I}[(\mathsf{F}_1, \ldots, \mathsf{F}_m)/(w_1, \ldots, w_m)]$

For a class $\mathcal{F}$ of frames,

$\mathsf{I}[\mathcal{F}] = \{\mathsf{I}[(\mathsf{F}_1, \ldots, \mathsf{F}_m)/(w_1, \ldots, w_m)] \mid \mathsf{F}_1, \ldots, \mathsf{F}_m \in \mathcal{F}\}$

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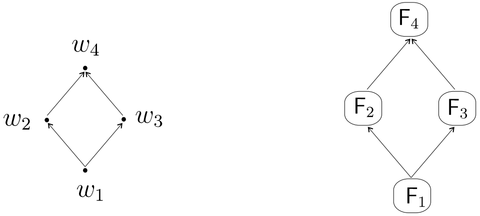

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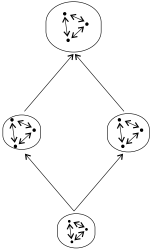
For a class $\mathcal{I}$ of finite partial orders,

$\mathcal{I}[\mathcal{F}] = \bigcup\{\mathsf{I}[\mathcal{F}] \mid \mathsf{I} \in \mathcal{I}\}$

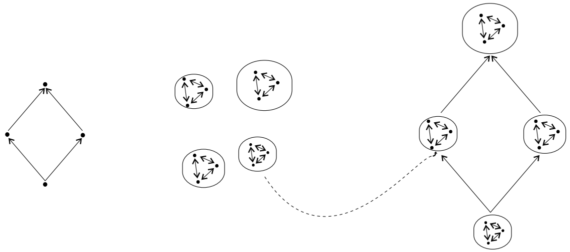
Ordinal sums of frames

Example: skeleton



Ordinal sums of frames

Example: skeleton



Every transitive frame can be considered as an ordinal sum of its clusters.

Ordinal sums of frames

Example: transitive frames

$\mathcal{PO}$ denotes the class of all finite (strict or non-strict) partial orders.

For $n \geq 1$, $\mathsf{C}_n = (W_n, W_n \times W_n)$, where $W_n = \{1, \ldots, n\}$;
C_0 denotes the irreflexive singleton $(\{0\}, \varnothing)$.

Let $\mathcal{F} = \{\mathsf{C}_0, \mathsf{C}_1, \mathsf{C}_2 \ldots\}$, $\mathcal{G} = \{\mathsf{C}_1, \mathsf{C}_2, \ldots\}$.
Then (up to isomorphisms):
$\mathcal{PO}[\mathcal{F}]$ is the class of all finite $\mathrm{K4}$-frames,
$\mathcal{PO}[\mathcal{G}]$ is the class of all finite $\mathrm{S4}$-frames.

Truth-preserving transformations for ordinal sums of frames

Treelike frames

$\mathcal{T}$ denotes the class of all finite transitive trees.

Lemma

Let $\mathcal{F}$ be a class of frames, I be a finite rooted partial order. Then for any $\mathsf{H} \in \mathsf{I}[\mathcal{F}]$ there exists a tree $\mathsf{T} \in \mathcal{T}$ such that for some $\mathsf{H}' \in \mathcal{T}[\mathcal{F}]$ we have $\mathsf{H}' \twoheadrightarrow \mathsf{H}$.

Corollary

φ is $\mathcal{PO}[\mathcal{F}]$-satisfiable $\Rightarrow$ φ is $\mathcal{T}[\mathcal{F}]$-satisfiable

Truth-preserving transformations for ordinal sums of frames

Restricting height and branching

$\langle\varphi\rangle$ denotes the cardinality of $Sub(\varphi)$.

Well-known fact: Any $\mathrm{K4}$-satisfiable formula φ is satisfiable in some finite frame with the height and the branching of its skeleton not more then $\langle\varphi\rangle$.

Truth-preserving transformations for ordinal sums of frames

Restricting height and branching

Let $\mathcal{T}_{n,b}$ denotes the class of transitive trees with the height not more then h and the branching not more then b:

$$\mathcal{T}_{h,b} = \{\mathsf{T} \in \mathcal{T} \mid Ht(\mathsf{T}) \leq h,\ Br(\mathsf{T}) \leq b\}.$$

Lemma

Let $\mathcal{F}$ be a class of transitive frames. If a formula φ is $\mathcal{PO}[\mathcal{F}]$-satisfiable, then φ is $\mathcal{T}_{\langle\varphi\rangle,\langle\varphi\rangle}[\mathcal{F}]$-satisfiable.

Selective filtration

A model $((W', R'), \theta')$ is a *weak submodel* of $((W, R), \theta)$, if
$W' \subseteq W, \quad R' \subseteq R$,
$\theta(p) = \theta'(p) \cap W'$ for any propositional variable p.

Definition

Let M be a model, Ψ a set of formulas closed under subformulas. A weak submodel M' of M is called a *selective filtration of* M *through* Ψ , if for any $w \in \mathrm{M}'$, for any formula ψ, we have

$$\Diamond\psi \in \Psi \;\&\; \mathrm{M}, w \vDash \Diamond\psi \;\Rightarrow\; \exists u \in R'(x)\; \mathrm{M}, u \vDash \psi,$$

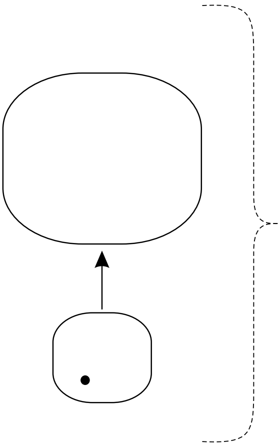
where R' is the accessability relation of M'.

Lemma

If M' *is a selective filtration of* M *through* Ψ*, then for any* $w \in \mathrm{M}'$*, for any* $\psi \in \Psi$*, we have*

$$\mathrm{M}, w \vDash \psi \Leftrightarrow \mathrm{M}', w \vDash \psi.$$

Big and small



Conditional satisfiability

M is a Kripke model

$$
\begin{array}{lcl}
\mathrm{M}, w \not\models \bot; & & \\
\mathrm{M}, w \models p & \Leftrightarrow & w \in \theta(p); \\
\mathrm{M}, w \models \varphi \to \psi & \Leftrightarrow & \mathrm{M}, w \not\models \varphi \text{ or } \mathrm{M}, w \models \psi; \\
\mathrm{M}, w \models \Diamond\varphi & \Leftrightarrow & \exists v (wRv \,\&\, \mathrm{M}, v \models \varphi).
\end{array}
$$

"φ is true at w in M".

Conditional satisfiability

M is a Kripke model, Ψ is a set of formulas

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\begin{array}{lcl}
(\mathsf{M}, w|\Psi) \not\models \bot & & \\
(\mathsf{M}, w|\Psi) \models p & \Leftrightarrow & w \in \theta(p); \\
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(\mathsf{M}, w|\Psi) \models \Diamond\varphi & \Leftrightarrow & \exists v(wRv \;\&\; \mathsf{M}, v \models \varphi) \\
 & & \text{or } \varphi \in \Psi \text{ or } \Diamond\varphi \in \Psi
\end{array}
$$

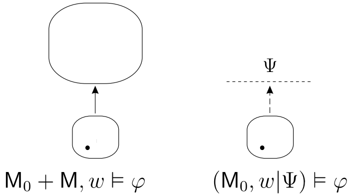
"φ *is true at* w *in* M *under the condition* Ψ ".

Consider transitive models M_0, M, their ordinal sum $\mathsf{M}_0 + \mathsf{M}$, and a formula φ. Put

$$\Psi = \{\psi \in Sub(\varphi) \mid \mathsf{M}, v \vDash \psi \text{ for some } v\}$$

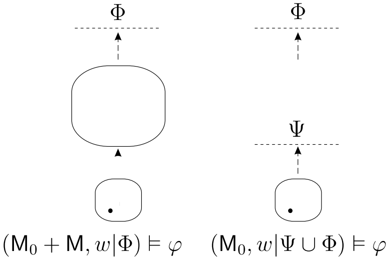
Then for any $w \in \mathsf{M}_0$,

$$\mathsf{M}_0 + \mathsf{M}, w \vDash \varphi \Leftrightarrow (\mathsf{M}_0, w|\Psi) \vDash \varphi$$



More generally: for any set of formulas Φ,

$$(\mathsf{M}_0{+}\mathsf{M}, w|\Phi) \vDash \varphi \Leftrightarrow (\mathsf{M}_0, w|\Psi{\cup}\Phi) \vDash \varphi$$



Moderate classes of frames

For a cone F, F $\Vdash \varphi$ means that φ is satisfiable at a root of F;
F $\mid \Psi \Vdash \varphi$ means $(\mathrm{M}, w|\Psi) \vDash \varphi$ for some model M based on F, where w is a root of F.
For a class of cones $\mathcal{F}$, $\mathcal{F} \mid \Psi \Vdash \varphi$ means that F $\mid \Psi \Vdash \varphi$ for some F $\in \mathcal{F}$

Definition

A sequence $(\mathcal{F}_n)_{n\in\mathbb{N}}$ of sets of rooted frames is called *d-moderate for $d \in \mathbb{N}$, if there exists an algorithm such that for any formula φ and any $\Psi, \Phi \subseteq Sub(\varphi)$ it decides whether*

$$\mathcal{F}_{\langle\varphi\rangle} \mid \Psi \Vdash \bigwedge \Phi$$

in space $O(\langle\varphi\rangle^d)$.

Moderate classes of frames: examples

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Example: $(\mathcal{F}_n)_{n\in\mathbb{N}}$ is moderate, if:

- $\mathcal{F}_n$ is the set of all (non-degenerate) clusters with cardinality not more then n:
  for all n $\mathcal{F}_n = \{\mathrm{C}_0, \dots, \mathrm{C}_n\}$ or for all n $\mathcal{F}_n = \{\mathrm{C}_1, \dots, \mathrm{C}_n\}$;
- $\mathcal{F}_n$ consists of a single frame which is a singleton:
  for all n $\mathcal{F}_n = \{\mathrm{C}_0\}$ or for all n $\mathcal{F}_n = \{\mathrm{C}_1\}$.

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$$\mathcal{F}_{\langle\varphi\rangle} \mid \Psi \Vdash \bigwedge \Phi$$

in space $O(\langle\varphi\rangle^d)$.

For classes $\mathcal{F}$, $\mathcal{G}$, put

$$\mathcal{F} + \mathcal{G} = \{\mathsf{F} + \mathsf{G} \mid \mathsf{F} \in \mathcal{F},\ \mathsf{G} \in \mathcal{G}\}$$

Lemma

If $(\mathcal{F}_n)_{n\in\mathbb{N}}$, $(\mathcal{G}_n)_{n\in\mathbb{N}}$ are moderate, then $(\mathcal{F}_n + \mathcal{G}_n)_{n\in\mathbb{N}}$ are moderate.

Main lemma

If $(\mathcal{F}_n)_{n\in\mathbb{N}}$ is d-moderate sequence of sets of cones, and P is a polynomial of degree d', then the sequence $(\mathcal{T}_{P(n),P(n)}[\mathcal{F}_n])_{n\in\mathbb{N}}$ is $\max\{2+d', d\}$-moderate.

Algorithm

Let `SatSimple`(φ, Φ, Ψ) decide whether $\mathcal{F} \mid \Psi \Vdash \bigwedge \Phi$ for any φ, $\Phi, \Psi \subseteq Sub(\varphi)$ in space $f(\langle\varphi\rangle)$.

Then the following algorithm decides whether $\mathcal{T}_{h,b}[\mathcal{F}] \mid \Psi \Vdash \bigwedge \Phi$ for any φ, $\Phi, \Psi \subseteq Sub(\varphi)$, $h, b > 0$ in space $O\left(f(\langle\varphi\rangle) + \langle\varphi\rangle bh\right)$:

Function `SatTree`$(\varphi; \Phi, \Psi; h, b)$ returns boolean;
Begin
  if `SatSimple`(φ, Φ, Ψ) then return(true);
  if $h > 1$ then
    for every integer b' such that $1 \leq b' \leq b$
      for every $\Phi_1, \ldots, \Phi_{b'} \subseteq Sub(\varphi)$
        if $\bigwedge_{1 \leq j \leq b'}$ `SatTree`$(\varphi, \Psi_j, \Psi, h-1, b)$ then
          if `SatSimple`$(\varphi, \Phi, \Psi \cup \Psi_1 \cdots \cup \Psi_{b'})$ then
            return(true);
  return(false);
End.

Algorithm

Lemma

Let $\mathcal{F}$ be a class of frames, and let $\mathsf{G} \in \mathcal{T}_{h+1,b}[\mathcal{F}]$ for some $h, b \geq 1$. Then G is either isomorphic to a frame $\mathsf{F} \in \mathcal{F}$ or isomorphic to a frame $\mathsf{F} + (\mathsf{G}_1 \sqcup \cdots \sqcup \mathsf{G}_{b'})$, where $1 \leq b' \leq b$, $\mathsf{F} \in \mathcal{F}$, $\mathsf{G}_1, \ldots \mathsf{G}_{b'} \in \mathcal{T}_{h,b}[\mathcal{F}]$.

Semantic condition

Theorem

Suppose that a logic L is characterized by $\mathcal{PO}[\mathcal{F}]$ for some class $\mathcal{F}$. If there exists a moderate sequence $(\mathcal{F}_n)_{n\in\mathbb{N}}$ such that $\mathcal{F}_n \subseteq \mathcal{F}$ for all $n \in \mathbb{N}$, and any L-satisfiable formula φ is $\mathcal{PO}[\mathcal{F}_{\langle\varphi\rangle}]$-satisfiable, then L is in PSPACE.

Corollary

If $\mathrm{L} = \mathrm{L}(\mathcal{PO}[\mathcal{F}])$ for some finite class of finite cones $\mathcal{F}$, then L is in PSPACE.

Examples

Consider the logics $\mathrm{K4}, \mathrm{S4}$, Gödel-Löb logic GL, and Grzegorczyk logic GRZ. They are well-known to be PSPACE-decidable. Let us illustrate, how this fact follows from the above theorem.

Examples

Consider the logics $\mathrm{K4}, \mathrm{S4}$, Gödel-Löb logic GL, and Grzegorczyk logic GRZ. They are well-known to be PSPACE-decidable. Let us illustrate, how this fact follows from the above theorem.

GRZ (GL) is the logic of all finite non-strict (strict) partial orders:
$\mathrm{GRZ} = \mathrm{L}(\mathcal{PO}[\{C_1\}]), \quad \mathrm{GL} = \mathrm{L}(\mathcal{PO}[\{C_0\}])$.
By the above corollary, GRZ and GL are in PSPACE.

Examples

Consider the logics $\mathrm{K4}, \mathrm{S4}$, Gödel-Löb logic GL, and Grzegorczyk logic GRZ. They are well-known to be PSPACE-decidable. Let us illustrate, how this fact follows from the above theorem.

Any $\mathrm{K4}$-satisfiable formula is satisfiable at some finite transitive frame F such that the cardinality of any cluster in F does not exceed $\langle\varphi\rangle$.
Put

$$\mathcal{F}_n^{\mathrm{K4}} = \{\mathsf{C}_0, \ldots, \mathsf{C}_n\}, \quad \mathcal{F}_n^{\mathrm{S4}} = \{\mathsf{C}_1, \ldots, \mathsf{C}_n\}.$$

Then for any φ we have:

$$\varphi \text{ is } \mathrm{K4}\text{-satisfiable iff } \varphi \text{ is } \mathcal{PO}[\mathcal{F}_{\langle\varphi\rangle}^{\mathrm{K4}}]\text{-satisfiable,}$$

$$\varphi \text{ is } \mathrm{S4}\text{-satisfiable iff } \varphi \text{ is } \mathcal{PO}[\mathcal{F}_{\langle\varphi\rangle}^{\mathrm{S4}}]\text{-satisfiable.}$$

Since the sequences $(\mathcal{F}_n^{\mathrm{K4}})_{n\in\mathbb{N}}$ and $(\mathcal{F}_n^{\mathrm{S4}})_{n\in\mathbb{N}}$ are moderate, then $\mathrm{K4}$ and $\mathrm{S4}$ are in PSPACE.

Example: logic LM

$\mathrm{LM} = \mathrm{K4} + \Diamond\top + \Diamond p_1 \wedge \Diamond p_2 \to \Diamond(\Diamond p_1 \wedge \Diamond p_2)$ (the logic of *interval inclusion*, *compact inclusion*, *chronological future*)

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For a transitive finite frame F,
$\mathsf{F} \vDash \mathrm{LM} \Leftrightarrow$ every its degenerate cluster has a unique successor C, and C is non-degenerate.

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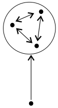

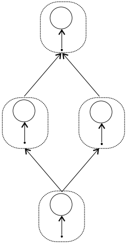

For $\mathcal{F} = \{\mathsf{C}_0 + \mathsf{C} \mid \mathsf{C} \text{ is a finite cluster}\}$,
$\mathcal{PO}[\mathcal{F}]$ is the class of all finite LM-frames (up to isomorphisms).

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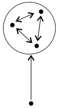

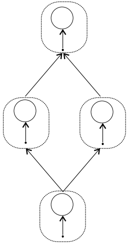

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LM has the FMP, thus: φ is LM-satisfiable $\Rightarrow$ φ is $\mathcal{PO}[\mathcal{F}_{\langle\varphi\rangle}]$-satisfiable, where $\mathcal{F}_n = \{\mathsf{C}_0 + \mathsf{C}_1, \ldots, \mathsf{C}_0 + \mathsf{C}_n\}$

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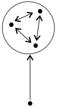

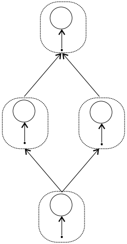

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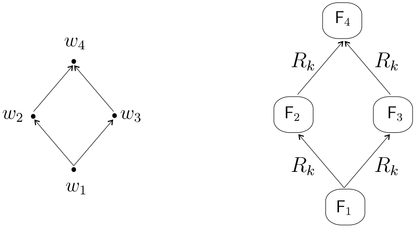
The sequence $(\mathcal{F}_n)_{n\in\mathbb{N}}$ is moderate, thus LM is PSPACE-decidable.

Multimodal case

Ordinal sums of multimodal frames

$\mathsf{I} = (W, R)$ is a finite partial order, $W = \{w_1, \ldots, w_n\}$
$\mathsf{F}_1 = (W_1, R_1^1, \ldots, R_N^1), \ldots, \mathsf{F}_n = (W_n, R_1^n, \ldots, R_N^n)$ are N-frames;
$1 \leq k \leq N$. $\mathsf{I}[k; (\mathsf{F}_1, \ldots, \mathsf{F}_m)/(w_1, \ldots, w_m)]$:



For a class $\mathcal{F}$ of N-frames, put

$$\mathsf{G}[k; \mathcal{F}] = \{\mathsf{I}[k; (\mathsf{F}_1, \ldots, \mathsf{F}_m)/(w_1, \ldots, w_m)] \mid \mathsf{F}_1, \ldots, \mathsf{F}_n \in \mathcal{F}\},$$

for a class $\mathcal{I}$ of finite partial orders, put

$$\mathcal{I}[k; \mathcal{F}] = \bigcup \{\mathsf{I}[k; \mathcal{F}] \mid \mathsf{I} \in \mathcal{G}\}.$$

Lemma

Let $\mathcal{F}$ be a class of N-frames, $1 \leq k \leq N$. If an N-formula φ is $\mathcal{PO}[k;\mathcal{F}]$-satisfiable, then φ is $\mathcal{T}_{\langle\varphi\rangle,\langle\varphi\rangle}[k;\mathcal{F}]$-satisfiable.

Definition

Let M be an N-model. A *condition for* M is a tuple $\overline{\Psi} = (\Psi_1, \ldots, \Psi_N)$ of sets of $N-$formulas. For an N-formula φ and a point $w \in$ M, we define the truth-relation $(\mathsf{M}, w|\overline{\Psi}) \vDash \varphi$ ("φ *is true at* w *in* M *under the condition* $\overline{\Psi}$"):

$$
\begin{array}{lcl}
(\mathsf{M}, w|\overline{\Psi}) \vDash p & \Leftrightarrow & \mathsf{M}, w \vDash p \\
(\mathsf{M}, w|\overline{\Psi}) \nvDash \bot & & \\
(\mathsf{M}, w|\overline{\Psi}) \vDash \varphi \to \psi & \Leftrightarrow & (\mathsf{M}, w|\overline{\Psi}) \nvDash \varphi \text{ or } (\mathsf{M}, w|\overline{\Psi}) \vDash \psi \\
 & & \\
(\mathsf{M}, w|\overline{\Psi}) \vDash \Diamond_k \varphi & \Leftrightarrow & \varphi \in \Psi_k \text{ or } \Diamond_k \varphi \in \Psi_k \text{ or} \\
 & & \text{for some } v \in R_k(w) \text{ we have } (\mathsf{M}, w|\Psi_k) \vDash \varphi,
\end{array}
$$

where $R_1, \ldots, R_N$ are the accessability relations in M.

An N-frame $\mathrm{G} = (W, R_1, \ldots, R_N)$ is called *rooted*, if for some $w \in W$ we have $\{w\} \cup R_1(w) \cup \cdots \cup R_N(w) = W$; w is called a *root* of G.
For an N-formula φ, put

$$Sub^*(\varphi) = Sub(\varphi) \cup \{\Diamond_i \psi \mid 1 \leq i, j \leq N, \quad \Diamond_j \psi \in Sub(\varphi)\}.$$

Definition

. A sequence $(\mathcal{F}_n)_{n \in \mathbb{N}}$ of sets of rooted N-frames is called *d-moderate*, if there exists an algorithm such that for any N-formula φ and any $\Phi, \Psi_1, \ldots, \Psi_n \subseteq Sub^*(\varphi)$, it decides whether

$$\mathcal{F}_{|Sub^*(\varphi)|} \mid (\Psi_1, \ldots, \Psi_n) \Vdash \bigwedge \Phi$$

in polynomial space.

Theorem

If $(\mathcal{F}_n)_{n\in\mathbb{N}}$ is d-moderate sequence of sets of rooted N-frames, $1 \leq k \leq N$, P is a polynomial of degree d', then the sequence

$$(\mathcal{T}_{P(n),P(n)}[k;\mathcal{F}_n])_{n\in\mathbb{N}}$$

is $\max\{2 + d', d\}$*-moderate.*

Japaridze's Polymodal Logic

For an N-frame $\mathsf{F} = (W, R_1, \dots, R_N)$ let

F_+ denote the (N+1)-frame $(W, \varnothing, R_1, \dots, R_N)$,

F_∞ denote the frame $(W, R_1, \dots, R_N, \varnothing, \varnothing, \dots)$
Hereditary strict orders:
$\mathcal{F}^{(1)}$ is the class of all finite strict partial orders;
$\mathcal{F}^{(N+1)} = \mathcal{PO}[1; \mathcal{G}^{(N)}]$, where $\mathcal{G}^N = \{\mathsf{F}_+ \mid \mathsf{F} \in \mathcal{F}^{(N)}\}$.
$\mathcal{F}^{\mathrm{J}} = \{\mathsf{F}_\infty \mid \mathsf{F} \in \mathcal{F}^{(N)} \text{ for some } N\}$.

Theorem (Beklemishev, 2007)

There exists a polynomial-time translation f such that for any formula φ we have

$$\mathrm{GLP} \vdash \varphi \Leftrightarrow \mathcal{F}^{\mathrm{J}} \models f(\varphi).$$

Let $\mathcal{T}^{(1)}_{h,b} = \mathcal{T}_{h,b}[\{C_0\}]$, i.e., $\mathcal{T}^{(1)}_{h,b}$ is the class (up to isomorphisms) of all finite transitive irreflexive trees with the height not more then h and the branching not more then b. Put

$$\mathcal{T}^{(N+1)}_{h,b} = \mathcal{T}_{h,b}[1; \{F_+ \mid F \in \mathcal{T}^{(N)}_{h,b}\}].$$

Corollary

If an N-modal formula φ is $\mathcal{F}^{\mathrm{J}}$-satisfiable, then φ is $\mathcal{T}^{(N)}_{N,N}$-satisfiable.

Theorem

The satisfiability problem for $\mathcal{F}^{\mathrm{J}}$ is in PSPACE.

Theorem

Japaridze's Polymodal Logic is PSPACE-decidable.