Advances in Modal Logic 2008
Lindström Theorems for Modal Logics

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Essential Definitions

Logic

A logic $\mathcal{L}$ is a pair $(L, \models_{\mathcal{L}})$

formulae closed under $\wedge, \neg$

$+$ natural assumptions such as
compatible with signature extensions/isomorphisms, etc.

$\mathcal{L} \leq \mathcal{L}'$

A logic $\mathcal{L}'$ is at least as expressive as $\mathcal{L}$ iff f.a. $\varphi \in \mathcal{L}$ there is $\varphi' \in \mathcal{L}'$ with the same semantics.
$\mathcal{L} \equiv \mathcal{L}'$ iff $\mathcal{L} \leq \mathcal{L}'$ and $\mathcal{L}' \leq \mathcal{L}$

compactness

$\text{sat}(T_0)$ for all finite $T_0 \subseteq T \Longrightarrow \text{sat}(T)$

relativisation

f.a. φ and unary P ex. φ^P expressing "φ true in the P-part"

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Lindström's Theorem (1969)

Theorem

$$FO \leq \mathcal{L}$$

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$$\Longrightarrow FO \equiv \mathcal{L}$$

Characterisation: Proper extensions of FO cannot preserve all properties

Lindström's Theorem (1969)

Idea

Assume $\varphi \in \mathcal{L} \backslash FO$

compactness: $\mathfrak{M} \equiv_{FO} \mathfrak{N}$, yet $\mathfrak{M} \models \varphi$ and $\mathfrak{N} \models \neg\varphi$

$\mathfrak{M} \cong_{part} \mathfrak{N}$, yet $\mathfrak{M} \models \varphi$ and $\mathfrak{N} \models \neg\varphi$

Lö-Sko: *obtain countable such* $\mathfrak{M}, \mathfrak{N}$

countable $\mathfrak{M}, \mathfrak{N}$ & $\mathfrak{M} \cong_{part} \mathfrak{N} \implies \mathfrak{M} \cong \mathfrak{N}$

Contradiction

Use of relativisation
allows to embed $\mathfrak{M}, \mathfrak{N}$ *together in one structure*

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Basic Modal Logic ML

$\mathrm{ML}(\tau, \Phi)$ ML-formulae over (τ, Φ)

modalities $\alpha \in \tau$: $\langle\alpha\rangle, [\alpha]$ modal operators
prop. letters $P \in \Phi$: P atomic formulae

Kripke semantics as usual

bisimulation — infinitary back&forth-games
$(\mathfrak{M}, w) \leftrightarrows (\mathfrak{N}, v)$ — $(\mathfrak{M}, w)$ bisimilar to $(\mathfrak{N}, v)$

n-bisimulation — game bounded to n rounds
$(\mathfrak{M}, w) \leftrightarrows_n (\mathfrak{N}, v)$ — $(\mathfrak{M}, w)$ n-bisimilar to $(\mathfrak{N}, v)$

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For finite τ and Φ $\quad \Longleftrightarrow$

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de Rijke's Approach to Lindström for ML

r-locality

$\varphi \in \mathcal{L}(\tau)$ is r-local if for all $(\mathfrak{M}, w)$

$$(\mathfrak{M}, w) \models \varphi \iff (\mathfrak{M}^r(w), w) \models \varphi$$

locality property

$\mathcal{L}$ has locality property

every $\mathcal{L}$-formula φ is r-local for some $r \in \mathbb{N}$.

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Lemma

In tree structures
$(\mathfrak{M}, w) \underline{\longleftrightarrow}_r (\mathfrak{N}, v) \iff (\mathfrak{M}^r(w), w) \underline{\longleftrightarrow} (\mathfrak{N}^r(w), v)$

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In tree structures and for finite (τ, Φ)

$(\mathfrak{M}, w) \equiv_r^{\mathrm{ML}} (\mathfrak{N}, v) \Longleftrightarrow (\mathfrak{M}^r(w), w) \Longleftrightarrow (\mathfrak{N}^r(w), v)$

de Rijke's Approach to Lindström for ML

Theorem

ML is maximally expressive among all logics with

- *compactness*
- *bisimulation invarian*
- *locality property*

Proof

We work with tree structures / finite (τ, Φ)

Assume $\varphi \in \mathcal{L} \backslash \mathrm{ML}$

compactness: $\varphi \dashv (\mathfrak{M}, w) \quad \equiv^{\mathrm{ML}} \quad (\mathfrak{N}, v) \models \neg\varphi$

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$\equiv^{\mathrm{ML}}_r$

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bisimulation invariance: Contradiction!

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FO Characterisation

- compactness
- Löwenheim-Skolem
- relativisation

ML Characterisation

- compactness
- bisimulation invariance
- locality

van Benthem's Revision

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Proposition

Any logic $\mathcal{L} \geq \mathrm{ML}$ *that is compact, bisimulation invariant and has relativisation has the locality property*

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Proposition

Any logic $\mathcal{L} \geq$ ML *that is compact, bisimulation invariant and has relativisation has the locality property*

But

locality argument does not carry over to ML[∀] and GF.
ML[∀] and GF are non-local.

ML[∀]

ML[∀]	$=$	ML + universal / global modality
$\overset{\forall}{\underline{\leftrightarrow}}$		bisimulation game with jumps

ML[∀]

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ML Characterisation

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ML[∀] Characterisation

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ML[∀]

$\mathrm{ML}[\forall] \;=\;$ ML + universal / global modality

$\overset{\forall}{\longleftrightarrow}$ bisimulation game with jumps

ML Characterisation

- compactness
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ML[∀] Characterisation

- compactness
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Assuming $\varphi \in \mathcal{L} \backslash \mathrm{ML}[\forall]$

- compactness

$$\varphi \dashv (\mathfrak{M}, w) \equiv_{\forall} (\mathfrak{N}, v) \models \neg\varphi$$

Question

How to derive $(\mathfrak{M}, w) \overset{\forall}{\longleftrightarrow} (\mathfrak{N}, v)$?

Types and Saturation

Hennessy-Milner-property

$\mathfrak{M}, \mathfrak{N}$ saturated: $(\mathfrak{M}, w) \equiv_{\forall} (\mathfrak{N}, v) \Longrightarrow (\mathfrak{M}, w) \overset{\forall}{\longleftrightarrow} (\mathfrak{N}, v)$.

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Types

- α-type at w f.a. $\Gamma_0 \subseteq \Gamma$ finite: $(\mathfrak{M}, w) \models \langle\alpha\rangle \bigwedge \Gamma_0$
- $\exists$-type f.a. $\Gamma_0 \subseteq \Gamma$ finite: $\mathfrak{M} \models \exists \bigwedge \Gamma_0$

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Realisation

- α-type at w ex. w' : $wR_\alpha w'$ and $(\mathfrak{M}, w') \models \Gamma$
- $\exists$-type ex. w' : $(\mathfrak{M}, w') \models \Gamma$

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Saturation

$\mathfrak{M}$ is saturated if $\forall \alpha \in \tau$ and $\forall w \in M$

- every α-type at w is realised at $(\mathfrak{M}, w)$
- every $\exists$-type of $\mathfrak{M}$ is realised in $\mathfrak{M}$.

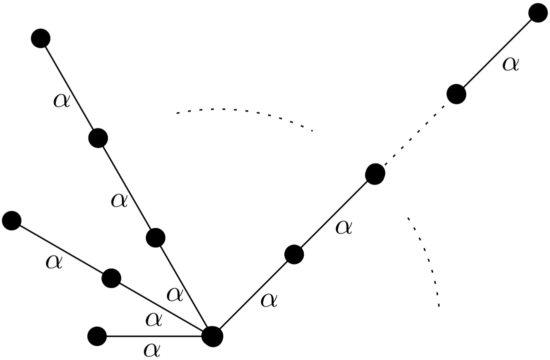

Figure: finitely deep branches of increasing depth

$\{\langle\alpha\rangle^n\top \mid n \in \mathbb{N}\}$ is an α-type of the root

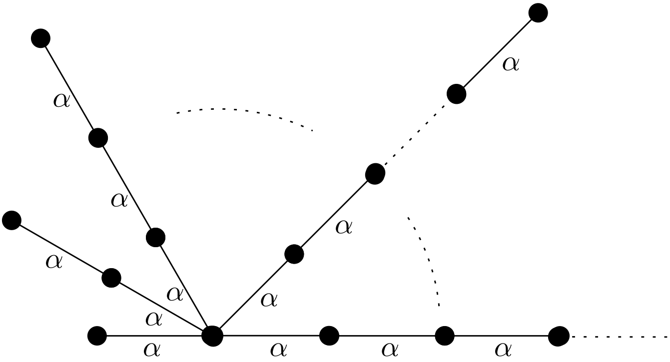

Figure: . . . saturated structure with additional infinite branch

$\{\langle\alpha\rangle^n \top \mid n \in \mathbb{N}\}$ is an α-type of the root

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Is it possible to obtain a saturated model?

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$\mathcal{L}$-Elementary Extension

$\mathfrak{M} \preccurlyeq_{\mathcal{L}} \mathfrak{M}'$ iff

- $\mathfrak{M} \hookrightarrow \mathfrak{M}'$
- $(\mathfrak{M}, w) \equiv_{\mathcal{L}} (\mathfrak{M}', w)$ for all $w \in M$

Tarski Union Property

A logic $\mathcal{L}$ has TUP iff

$$\mathfrak{M}_0 \preccurlyeq_{\mathcal{L}} \mathfrak{M}_1 \preccurlyeq_{\mathcal{L}} \mathfrak{M}_2 \preccurlyeq_{\mathcal{L}} \mathfrak{M}_3 \ldots$$

$$\mathfrak{M}^* := \bigcup_{i<\omega} \mathfrak{M}_i$$

$$\Longrightarrow \text{f.a. } i: \quad \mathfrak{M}_i \preccurlyeq_{\mathcal{L}} \mathfrak{M}^*$$

Coding an $\mathcal{L}$-elementary extension

By bisimulation invariance of $\mathcal{L}$, we may assume $\mathfrak{M}$ is a forest structure

- Add new marker predicates P_w to (τ, Φ) f.a. $w \in M$
- Find $\mathcal{L}$-formulae such that

$$\mathfrak{N} \models \Theta \Longrightarrow \mathfrak{M} \preccurlyeq_{\mathcal{L}} \mathfrak{N}$$

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$\exists P_w$	$w \in M$
$\forall(P_w \longrightarrow \neg P_{w'})$	$w \neq w'$
$\forall(P_w \longrightarrow \langle\alpha\rangle P_{w'})$	$w R_\alpha w'$
$\forall(P_w \longrightarrow \neg\langle\alpha\rangle P_{w'})$	not $w R_\alpha w'$
$\forall(P_w \longrightarrow \varphi)$	$(\mathfrak{M}, w) \models \varphi$
	$\varphi \in \mathcal{L}(\tau, \Phi)$

Realising Types

More marker predicates for the desired realisation of types

- Γ α-type of w: $P_w^{\alpha,\Gamma}$
- Γ $\exists$-type: P_Γ

Add new formulae to Θ

$$\forall(P_w \longrightarrow \langle\alpha\rangle P_w^{\alpha,\Gamma})$$

$$\exists P_\Gamma$$

$$\forall(Q \longrightarrow \varphi) \qquad Q = P_w^{\alpha,\Gamma}, P_\Gamma \quad \text{and} \quad \varphi \in \Gamma$$

Constructing a saturated model

Let $\mathfrak{M}$ be a forest structure, then for any $\mathfrak{N} \models \Theta$

- $\mathfrak{M} \preccurlyeq_{\mathcal{L}} \mathfrak{N}$
- $(\mathfrak{N}, w)$ realises all types of $(\mathfrak{M}, w)$

Θ is satisfiable.

Proposition

Let $\mathcal{L} \geq \mathrm{ML}[\forall]$ and have

- *compactness*
- *bisimulation invariance*
- *TUP*

then for all forest structures $\mathfrak{M}$ there is $\mathfrak{M}^*$ saturated with $\mathfrak{M} \preccurlyeq_{\mathcal{L}} \mathfrak{M}^*$

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Proof of Lindström for ML[∀]

Assume $\varphi \in \mathcal{L} \backslash \mathrm{ML}[\forall]$

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- TUP $\varphi \dashv (\mathfrak{M}^*, w) \equiv_\forall (\mathfrak{N}^*, v) \models \neg\varphi$
- bisimulation invariance $(\mathfrak{M}^*, w) \models \varphi \iff (\mathfrak{N}^*, v) \models \varphi$

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- TUP $\quad \varphi \dashv (\mathfrak{M}^*, w) \equiv_\forall (\mathfrak{N}^*, v) \models \neg\varphi$
- bisimulation invariance $\quad (\mathfrak{M}^*, w) \models \varphi \Longleftrightarrow (\mathfrak{N}^*, v) \models \varphi$

Proposition (Hennessy-Milner)

$\mathfrak{M}^*, \mathfrak{N}^*$ *saturated:*

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Summary

ML Characterisation

- compactness
- bisimulation invariance
- relativisation

ML[∀] Characterisation

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Still open:

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Under bisimulation invariance and compactness

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Counter examples for relativisation must break TUP