Relational syllogistic logics

and other connections between modal logic and natural logic

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The talk is based on recent work with
Ian Pratt-Hartmann

A quiz

Which branch of logic features all of the following?

- ⋆ Ancient roots yet contemporary applications.
- ⋆ A plethora of different logical systems.
- ⋆ Connections to algebraic semantics.
- ⋆ Interesting proof theory.
- ⋆ Decidability/complexity results.

A quiz

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Answers: modal logic and syllogistic logic.

My goal in this talk is to introduce you to syllogistic logic and to make the case for all of these points.

The simplest fragment "of all"

It is just the sentences

$$\textit{All } X \textit{ are } Y$$

where X and Y are variables (for plural nouns).
The semantics uses models $\mathcal{M}$ which start with an underlying universe M and include for each X a subset

$$X^{\mathcal{M}} \subseteq M.$$

The semantics

$$\mathcal{M} \models \textit{All } X \textit{ are } Y \quad \text{iff} \quad X^{\mathcal{M}} \subseteq Y^{\mathcal{M}}.$$

Proof Rules for *All*

$$\frac{}{\textit{All } X \textit{ are } X} \qquad \frac{\textit{All } X \textit{ are } Z \quad \textit{All } Z \textit{ are } Y}{\textit{All } X \textit{ are } Y}$$

The usual definitions

$\Gamma \models S$

$\Gamma \vdash S$

soundness and completeness of the resulting logical system

Proof of completeness

Suppose that $\Gamma \models$ *All X are Y*.
Let M be the set of variables.
Define $A \leq B$ to mean that $\Gamma \vdash$ *All A are B*.
Check that this is reflexive and transitive, using the logic.
The semantics is via downsets:

$$A^{\mathcal{M}} \quad = \quad \downarrow A \quad = \quad \{B : B \leq A\}.$$

By transitivity, $\mathcal{M} \models \Gamma$.

This is the simplest completeness theorem in logic

Suppose that $\Gamma \models \textit{All } X \textit{ are } Y$.
Let M be the set of variables.
Define $A \leq B$ to mean that $\Gamma \vdash \textit{All } A \textit{ are } B$.
The semantics is $A^{\mathcal{M}} = \downarrow A$.
Then $\mathcal{M} \models \Gamma$.
That is, $X^{\mathcal{M}} \subseteq Y^{\mathcal{M}}$.
But by reflexivity, $X \in X^{\mathcal{M}}$.
And so $X \in Y^{\mathcal{M}}$; this means that $X \leq Y$.
And this means exactly what we want: $\Gamma \vdash \textit{All } X \textit{ are } Y$.

Proof Rules for *All* and *Some*

$$\frac{}{\textit{All } X \textit{ are } X} \qquad \frac{\textit{All } X \textit{ are } Z \quad \textit{All } Z \textit{ are } Y}{\textit{All } X \textit{ are } Y}$$

$$\frac{\textit{Some } X \textit{ are } Y}{\textit{Some } Y \textit{ are } X} \qquad \frac{\textit{Some } X \textit{ are } Y}{\textit{Some } X \textit{ are } X}$$

$$\frac{\textit{All } Y \textit{ are } Z \quad \textit{Some } X \textit{ are } Y}{\textit{Some } X \textit{ are } Z}$$

Fragments which have been treated

I won't talk about any of these

(i) the fragment with *All X are Y*
(ii) the fragment with *Some X are Y*
(iii) = (i)+(ii)
(iv) = (iii) + sentences involving proper names
(v) = (i) + *No X are Y*
(vi) *All* + Some + *No*
(vii)= (vi) + Names
(viii) boolean combinations of (vii)
(ix)= (i) + There are at least as many *X* as *Y*
(x)= boolean combinations of (ix) + Some + No
(xi) *All X which are Y are Z*
(xii) Most *X* are *Y*
(xiii) Most + *Some*

First-order logic is too big and too small for natural language.

The languages $\mathcal{S}$ and $\mathcal{S}^\dagger$

The next step is to add a complementation operation to nouns.
We understand X'' to be the same as X.
The semantics is $(X')^{\mathcal{M}} = M \setminus X^{\mathcal{M}}$.

$$\left.\begin{array}{ll} \textit{All } X \textit{ are } Y & \textit{Some } X \textit{ are } Y' \\ \textit{All } X \textit{ are } Y' & \textit{Some } X \textit{ are } Y \\ \textit{All } X' \textit{ are } Y' & \textit{Some } X' \textit{ are } Y \\ \hline \textit{All } X' \textit{ are } Y & \textit{Some } X' \textit{ are } Y' \end{array}\right\} \mathcal{S}^\dagger$$

Read All X are Y' as either

All X are non-Y or as No X are Y

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In $\mathcal{S}$, every sentence is semantically equivalent to a sentence where the subject NP is positive:

$$\begin{array}{lcl} \textit{All } X' \textit{ are } Y' & \equiv & \textit{All } Y \textit{ are } X \\ \textit{Some } X' \textit{ are } Y & \equiv & \textit{Some } Y \textit{ are } X' \end{array}$$

This is not the case for $\mathcal{S}^\dagger$.

$\mathcal{S}$ AND $\mathcal{S}^{\dagger}$ ARE CLOSED UNDER SEMANTIC NEGATION

Sentence φ	its negation $\overline{\varphi}$
All X are Y	*Some X are Y′*
All X are Y′	*Some X are Y*
All X′ are Y′	*Some X′ are Y*
All X′ are Y	*Some X′ are Y′*

The system $\mathsf{S}^{\dagger}$ for $\mathcal{S}^{\dagger}$

$$\frac{}{\textit{All } X \textit{ are } X} \qquad \frac{\textit{Some } X \textit{ are } Y}{\textit{Some } X \textit{ are } X} \qquad \frac{\textit{Some } X \textit{ are } Y}{\textit{Some } Y \textit{ are } X}$$

$$\frac{\textit{All } X \textit{ are } Z \quad \textit{All } Z \textit{ are } Y}{\textit{All } X \textit{ are } Y}\ \textit{Barbara}$$

$$\frac{\textit{All } Y \textit{ are } Z \quad \textit{Some } X \textit{ are } Y}{\textit{Some } X \textit{ are } Z}\ \textit{Darii}$$

$$\frac{\textit{All } Y \textit{ are } Y'}{\textit{All } Y \textit{ are } X}\ \textit{Zero} \qquad \frac{\textit{All } Y' \textit{ are } Y}{\textit{All } X \textit{ are } Y}\ \textit{One}$$

$$\frac{\textit{All } X \textit{ are } Y'}{\textit{All } Y \textit{ are } X'}\ \textit{Antitone} \qquad \frac{\textit{Some } X \textit{ are } X'}{S}\ \textit{Ex falso quodlibet}$$

A FORMAL PROOF TREE IN OUR S†

Let Γ be

$$\{\textit{All } B \textit{ are } X, \textit{All } B' \textit{ are } X, \textit{All } Y \textit{ are } C, \textit{Some } A \textit{ are } C'\}$$

Here is a proof tree showing $\Gamma \vdash \textit{Some } X \textit{ are } Y'$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\textit{All } B \textit{ are } X}{\textit{All } X' \textit{ are } B'} \quad \textit{All } B' \textit{ are } X}{\textit{All } X' \textit{ are } X}}{\textit{All } Y' \textit{ are } X} \qquad \dfrac{\dfrac{\dfrac{\textit{All } Y \textit{ are } C}{\textit{All } C' \textit{ are } Y'} \quad \dfrac{\textit{Some } A \textit{ are } C'}{\textit{Some } C' \textit{ are } C'}}{\textit{Some } C' \textit{ are } Y'}}{\textit{Some } Y' \textit{ are } Y'}}{\textit{Some } Y' \textit{ are } X}}{\textit{Some } X \textit{ are } Y'}$$

Orthoposets

Definition

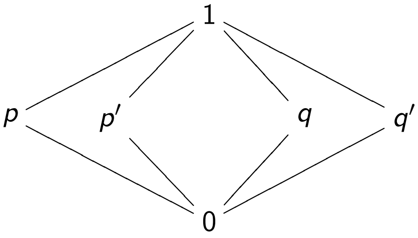
An orthoposet is a tuple $(P, \leq, 0, ')$ such that

- **poset** $\leq$ is a reflexive, transitive, and antisymmetric relation on the set P.
- **zero** $0 \leq p$ for all $p \in P$.
- **antitone** If $x \leq y$, then $y' \leq x'$.
- **involutive** $x'' = x$.
- **inconsistency** If $x \leq y$ and $x \leq y'$, then $x = 0$.

A Key Point

Orthoposets need not have a meet or join operation.

The Chinese lantern M_2



Here and elsewhere, we understand $(x')' = x$, $0' = 1$, $1' = 0$.

Orthoposets

Example

For all sets X we have an orthoposet $(\mathcal{P}(X), \subseteq, \emptyset, ')$, where $a' = X \setminus a$ for all subsets a of X.

Orthoposets from the logic

Let Γ be any set of sentences in the fragment.
Let $\mathcal{V}$ be the set of variables.
We already know the preorder $\leq$:

$$X \leq \mathrm{Y} \quad \text{iff} \quad \Gamma \vdash \textit{All } X \textit{ are } Y.$$

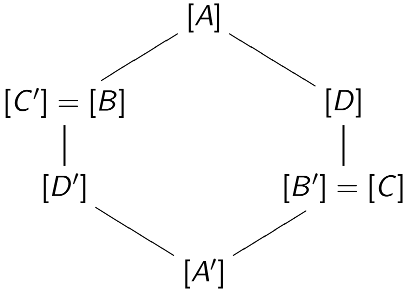
(Some plays no role.)
We have an induced equivalence relation $\equiv$,
and we take $\mathcal{V}_\Gamma$ to be the quotient $\mathcal{V}/\!\equiv$.
We might need to add a fresh 0 to $\mathcal{V}/\!\equiv$.
It is not hard to check that we have an orthoposet $\mathcal{V}_\Gamma$.

Orthoposets from logic, concretely

Let $\Gamma =$
$\{$All B are A, All B' are A, All C' are B, All C are B', All C are $D\}$.
Then

$$\begin{array}{ll} [A] = \{A\} & [A'] = \{A'\} \\ [B] = \{B, C'\} & [B'] = \{B', C\} \\ [C] = \{B', C\} & [C'] = \{B, C'\} \\ [D] = \{D\} & [D'] = \{D'\} \end{array}$$

Here is a picture of the orthoposet $\mathcal{V}_\Gamma$:



Points of orthoposets

A point of a orthoposet $P = (P, \leq, 0, ')$ is a subset $\mathcal{S} \subseteq P$ with the following properties:

UP-CLOSED If $p \in S$ and $p \leq q$, then $q \in \mathcal{S}$.

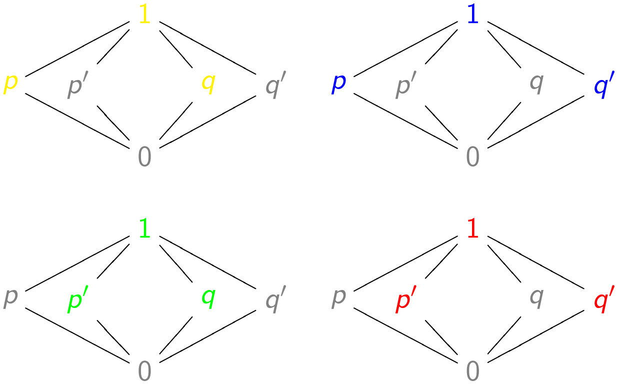
COMPLETE For all p, either $p \in S$ or $p' \in \mathcal{S}$.

PAIRWISE COMPATIBLE For all $p, q \in \mathcal{S}$, $p \not\leq q'$.

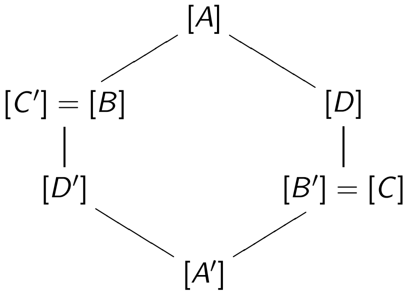
Points are sets

Look back at the Chinese lantern.

There are four points here: the sets marked ●, ●, ●, and ●:

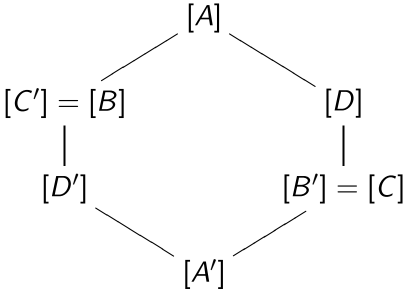


What are the points?



What are the points?

There are three points.



$\mathcal{S} = \{[D'], [B], [A]\}$, $\mathcal{T} = \{[B'], [D], [A]\}$, $\mathcal{U} = \{[B], [D], [A]\}$.

Pairwise consistent sets extend to points

the usual Zorn's Lemma argument

Lemma

For a subset S_0 of an orthoposet $P = (P, \leq, 0, ')$, the following are equivalent:

1. *S_0 is a subset of a point $\mathcal{S}$ in P.*
2. *S_0 is pairwise compatible: $(\forall p, q \in \mathcal{S}) p \not\leq q'$.*

Let $P = (P, \leq, ')$ be an orthoposet.
Let $points(P)$ be the set of points of P.
We have an orthoposet

$$(\mathcal{P}(points(P)), \subseteq, \emptyset, ')$$

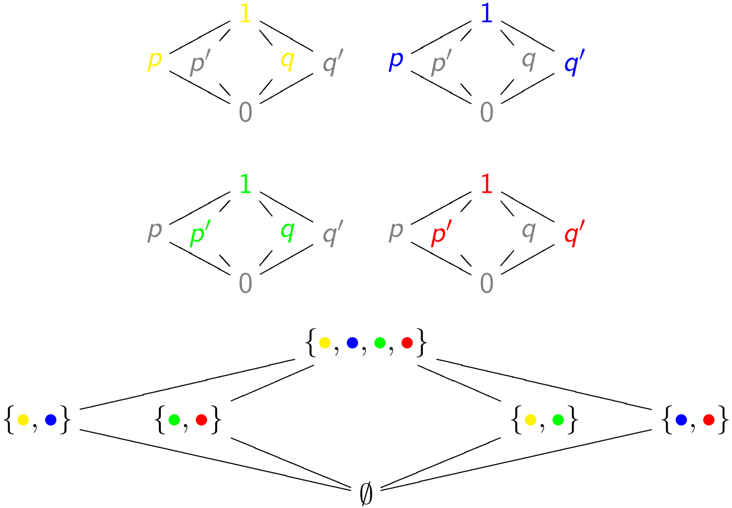
Let $m : P \to \mathcal{P}(points(P))$ be given by

$$m(p) \quad = \quad \{\mathcal{S} : p \in \mathcal{S}\}.$$

Theorem

m is a strict morphism of orthoposets:
$m(0) = \emptyset$,
$m(p') = (m(p))'$,
and $p \leq q$ *if and only if* $m(p) \subseteq m(q)$.

How the representation works



The canonical model

Lemma

Let Γ *be consistent in* $\mathcal{L}(\mathsf{all}, \mathsf{some}, ')$.
There is a canonical model $\mathcal{M} = (M, \cdot^{\mathcal{M}})$ *such that*

1. $\mathcal{M} \models \Gamma$.
2. *If* $\mathcal{M} \models$ *All X are Y, then* $\Gamma \vdash$ *All X are Y.*

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Proof.

Let $\mathcal{V}_\Gamma$ be the syntactic orthoposet for Γ. Let $M = points(\mathcal{V}_\Gamma)$.
The interpretation $(\cdot)^{\mathcal{M}} : \mathcal{V} \to \mathcal{P}(M)$ is given by

$$\mathcal{V} \xrightarrow{n} \mathcal{V}_\Gamma \xrightarrow{m} \mathcal{P}(points(\mathcal{V}_\Gamma)) = \mathcal{P}(M)$$

Key point If Γ contains Some U are V, need a point including $\{[\mathrm{U}], [V]\}$.
If none exists, then wlog $\mathrm{U} \leq V'$. But then Γ is inconsistent. □

The canonical model

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There is a canonical model $\mathcal{M} = (M, \cdot^{\mathcal{M}})$ such that

1. $\mathcal{M} \models \Gamma$.
2. *If $\mathcal{M} \models$ All X are Y, then $\Gamma \vdash$ All X are Y.*

This is half of the work in the completeness theorem for the logic. More has to be done, to show that if Γ is consistent and $\Gamma \models$ Some X are Y, then $\Gamma \vdash$ Some X are Y.

Earlier sources for the Representation Theorem

N. Zierler and M. Schlessinger

Boolean embeddings of orthomodular sets and quantum logic.
Duke Mathematical Journal 32 (1965), 251–262.

F. Katrnoška

On the representation of orthocomplemented posets.
Comment. Math. Univ. Carolinae 23 (1982), 489–498.

C. S. Calude, P. H. Hertling, K. Svozil

Embedding quantum universes into classical ones.
Foundations of Physics, 29, 3 (1999), 349-379.

The new work of this talk begins here

Augustus de Morgan famously observed
that the Aristotelian syllogistic
cannot account for the validity of even th e most elementary
inferences involving relational facts, for example (de Morgan 1847):

$$\frac{\textit{Every man is an animal}}{\textit{He who kills a man kills an animal}}$$

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$$\frac{\textit{Every man is an animal}}{\textit{He who kills a man kills an animal}}$$

What do you think about this one?

$$\frac{\textit{All skunks are mammals}}{\textit{All who fear all who respect all skunks fear all who respect all mammals}}$$

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$$\frac{\textit{Every man is an animal}}{\textit{He who kills a man kills an animal}}$$

It follows, using an interesting monotonicity principle

$$\frac{\dfrac{\textit{All skunks are mammals}}{\textit{All who respect all mammals respect all skunks}}}{\textit{All who fear all who respect all skunks fear all who respect all mammals}}$$

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An easier one:

$$\frac{\textit{All } X \textit{ see all } Y}{\textit{All } X \textit{ see some } Y}$$

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Augustus de Morgan famously observed
that the Aristotelian syllogistic
cannot account for the validity of even th e most elementary
inferences involving relational facts, for example (de Morgan 1847):

Every man is an animal

He who kills a man kills an animal

What do you think about this one?

All xenophobics hate all actors
All yodelers hate all zookeepers
All non-yodelers hate all non-actors
All wardens are xenophobics

?? All wardens hate all zookeepers

$\mathcal{L}(all, \ ', \text{VERBS})$, USING *see* FOR THE VERB

$$\frac{}{All\ X\ are\ X} \qquad \text{All } X^{\downarrow} \text{ are } Y^{\uparrow}$$

$$\frac{All\ Y\ are\ Y'}{All\ Y\ are\ X}\ Zero \qquad \frac{All\ Y'\ are\ Y}{All\ X\ are\ Y}\ One \qquad \frac{All\ X\ are\ Y'}{All\ Y\ are\ X'}\ Antitone$$

$$All\ X^{\downarrow}\ see\ all\ Y^{\downarrow}$$

$$\frac{All\ X\ see\ all\ Y \quad All\ X'\ see\ all\ Y}{All\ Z\ see\ all\ Y}\ LEM_{subject}$$

$$\frac{All\ X\ see\ all\ Y \quad All\ X\ see\ all\ Y'}{All\ X\ see\ all\ Z}\ LEM_{object}$$

$$\frac{All\ X\ see\ all\ A \quad All\ Y\ see\ all\ Z \quad All\ Y'\ see\ all\ A'}{All\ X\ see\ all\ Z}\ 3pr$$

We also need the dual form of (3pr).

Example

$$\frac{\begin{array}{l}\textit{All xenophobics hate all actors}\\ \textit{All yodelers hate all zookeepers}\\ \textit{All non-yodelers hate all non-actors}\\ \textit{All wardens are xenophobics}\end{array}}{\textit{All wardens hate all zookeepers}}$$

$$\dfrac{\dfrac{\textit{All } X \textit{ hate all } A \quad \textit{All } Y \textit{ hate all } Z \quad \textit{All } Y' \textit{ hate all } A'}{\textit{All } X \textit{ hate all } Z}\,\textit{3pr} \qquad \textit{All } W \textit{ are } X}{\textit{All } W \textit{ hate all } Z}$$

It does seem crazy to take (3pr) as a single rule.
But the fact is that it cannot be simplified:
no pair of its premises have any non-trivial consequences in this language.

The Canonical Model, again

We start with a consistent set Γ.
We again construct the syntactic orthoposet $\mathcal{V}_\Gamma$.
Let M be the set of all points on $\mathcal{V}_\Gamma$.
Let $X^{\mathcal{M}} = \{\mathcal{S} \in M : X \in \mathcal{S}\}$, and let

$$see^{\mathcal{M}} \quad = \quad \{(\mathcal{S}, \mathcal{T}) : (\exists A \in \mathcal{S})(\exists B \in \mathcal{T})\ \Gamma \vdash \text{All } A \text{ } see \text{ all } B\}.$$

The Canonical Model, again

We start with a consistent set Γ.
We again construct the syntactic orthoposet $\mathcal{V}_\Gamma$.
Let M be the set of all points on $\mathcal{V}_\Gamma$.
Let $X^{\mathcal{M}} = \{\mathcal{S} \in M : X \in \mathcal{S}\}$, and let

$$see^{\mathcal{M}} \quad = \quad \{(\mathcal{S}, \mathcal{T}) : (\exists A \in \mathcal{S})(\exists B \in \mathcal{T})\ \Gamma \vdash \text{All } A \textit{ see } \text{all } B\}.$$

Lemma

Fix Γ, and also fix X and Y such that

$$\Gamma \nvdash \textit{All } X \textit{ see all } Y.$$

Then there are points $\mathcal{S}^*$ and $\mathcal{T}^*$ such that $X \in \mathcal{S}^*$, $Y \in \mathcal{T}^*$, and for all $A \in \mathcal{S}^*$ and $B \in \mathcal{T}^*$, $\Gamma \nvdash$ All A see all B.

This would complete the proof, but this lemma is very long!

End of digression: back to verbs

We now want to formulate systems with All and Some,
nouns (allowing complementation)
and also verbs (now also allowing complementation).
Eventually we'll want relative clauses, but this will wait.

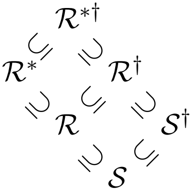
Most of the rest of this talk reports on joint
with Ian Pratt-Hartmann.

End of digression: back to verbs

We now want to formulate systems with All and Some,
nouns (allowing complementation)
and also verbs (now also allowing complementation).
Eventually we'll want relative clauses, but this will wait.
Issues:

- ⋆ Contradictions: Ex falso quodlibet, or reductio ad absurdum?
- ⋆ Quantifier scope ambiguity: All dogs see some cat
- ⋆ Relative clauses without committing to a syntax.

THE SYSTEMS



$\mathcal{S}$	classical syllogistic
$\mathcal{S}^{\dagger}$	syllogistic with full noun negation
$\mathcal{R}$	relational syllogistic
$\mathcal{R}^{\dagger}$	relational syllogistic with noun-negations
$\mathcal{R}^{*}$	relational syllogistic allowing subject NPs to be relative clauses
$\mathcal{R}^{*\dagger}$	relational syllogistic allowing subject NPs to be relative clauses and full noun-negation

The language $\mathcal{R}$

$\mathcal{R}$ has all sentences of the form

All/some X (don't) see all/some Y
All/some X are/aren't Y

We do not use complemented noun variables in this fragment.
Just as in description logic, the semantics first considers expressions like

see all Y	*don't see all Y*
see some Y	*don't see all Y*

as set expressions, and then uses the natural semantics.
Then we read our syntax above as

All/some X (see all/some Y)
All/some X ((don't) see all/some Y)

The language $\mathcal{R}$, more explicitly

All X are Y	*All X aren't Y* $\equiv$ *No X are Y*
Some X are Y	*Some X aren't Y*
All X see all Y	*All X don't see all Y* $\equiv$ *No X sees any Y*
All X see some Y	*All X don't see some Y* $\equiv$ *No X sees all Y*
Some X see all Y	*Some X don't see any Y*
Some X see some Y	*Some X don't see some Y*

The interpretation is the natural one, using the subject wide scope readings in the ambiguous cases.
Once again, this is $\mathcal{R}$.
The language $\mathcal{R}^{\dagger}$ has complemented variables X' on top of $\mathcal{R}$.

There are no finite, sound, and complete syllogistic logics $\vdash_{\mathcal{X}}$ for $\mathcal{R}$

Suppose towards a contradiction that $\mathcal{X}$ did it.
We allow rules with arbitrarily many premises.
Fix $n \in \mathsf{N}$ greater than the number of premises in any rule in $\mathcal{X}$.
Let $Y_1, \ldots, Y_n$ be distinct variables .
Let Γ be the following set of $\mathcal{R}$-formulas:

All Y_i see some Y_{i+1}	$(1 \leq i < n)$
All Y_1 see all Y_n	
All Y_i are Y_i	$(1 \leq i < n)$
All Y_i aren't Y_j	$(1 \leq i < j \leq n)$

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All Y_i *see some* Y_{i+1}	$(1 \leq i < n)$
All Y_1 *see all* Y_n	
All Y_i *are* Y_i	$(1 \leq i < n)$
All Y_i *aren't* Y_j	$(1 \leq i < j \leq n)$

Observe that $\Gamma \models$ *All* Y_1 *see some* Y_n,
but this sentence is not in Γ.

Proof, continued

All Y_i see some Y_{i+1}	$(1 \leq i < n)$
All Y_1 see all Y_n	
All Y_i are Y_i	$(1 \leq i < n)$
All Y_i aren't Y_j	$(1 \leq i < j \leq n)$

For $1 \leq i < n$, let $\Delta_i = \Gamma \setminus \{$*All Y_i see some Y_{i+1}*$\}$.

Claim If $\varphi \in \mathcal{R}$ and $\Delta_i \models \varphi$, then $\varphi \in \Gamma$.

Proof, continued

All Y_i see some Y_{i+1}	$(1 \leq i < n)$
All Y_1 see all Y_n	
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For $1 \leq i < n$, let $\Delta_i = \Gamma \setminus \{$*All Y_i see some Y_{i+1}*$\}$.

Claim If $\varphi \in \mathcal{R}$ and $\Delta_i \models \varphi$, then $\varphi \in \Gamma$.

It follows this claim that $\Gamma \not\vdash_{\mathcal{X}} \gamma$.

Reason: No rule of $\mathcal{X}$ has more than $n - 1$ premises.

Any instance of those premises contained in Γ must be contained in Δ_i for some i.

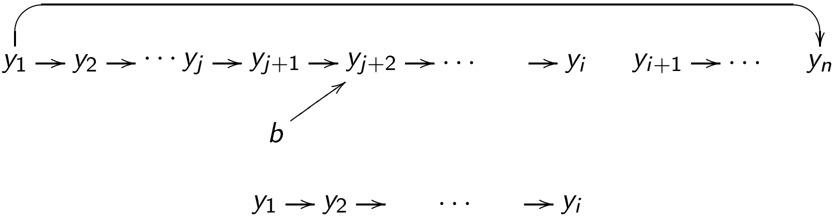
By induction, we see that Γ is closed under deduction.

So the logic is complete.

The models

$$\begin{array}{ll} \textit{All } Y_i \textit{ see some } Y_{i+1} & (1 \leq i < n) \\ \textit{All } Y_1 \textit{ see all } Y_n & \\ \textit{All } Y_i \textit{ are } Y_i & (1 \leq i < n) \\ \textit{All } Y_i \textit{ aren't } Y_j & (1 \leq i < j \leq n) \end{array}$$

For $1 \leq i < n$, let $\Delta_i = \Gamma \setminus \{$*All* Y_i *see some* $Y_{i+1}\}$.
Proof sketch Every sentence in the entire language is either in Γ, or is falsified in one of the following models:



What to do?

We should not be deterred by the negative result:
there are two ways to go:

1. Move from a pure syllogistic system to a more liberal type of logic.
2. Use infinitely many rules.

Refutation completeness

Γ is consistent in a syllogistic system $\vdash_{\mathcal{X}}$ if

$$\Gamma \nvdash_{\mathcal{X}} \textit{Some } A \textit{ are } A'.$$

A syllogistic system $\vdash_{\mathcal{X}}$ is refutation complete if every consistent set is satisfiable.
Adding reductio ad absurdum (RAA) to a refutation complete system $\vdash_{\mathcal{X}}$ renders it complete.

$$\dfrac{\begin{array}{c}\cdots[\bar{\varphi}]\;\;\cdots[\bar{\varphi}]\cdots \\ \vdots \\ \textit{Some } X \textit{ are } X'\end{array}}{\varphi}\;(\text{RAA}).$$

In a refutation complete system, every assertion $\Gamma \vdash_{\mathcal{X}} \varphi$ has a derivation in which RAA is used only at the end.

A refutation-complete system **R** for $\mathcal{R}$

On top of our system for $\mathcal{S}$, and one rule is missing from the list

$$\text{All } X^{\downarrow} \text{ (don't) see all } Y^{\downarrow}$$
$$\text{Some } X^{\uparrow} \text{ (don't) see all } Y^{\downarrow}$$
$$\text{All } X^{\downarrow} \text{ (don't) see some } Y^{\uparrow}$$
$$\text{Some } X^{\uparrow} \text{ (don't) see some } Y^{\uparrow}$$

$$\frac{\text{All } X \text{ aren't } X}{\text{All } X \text{ see all } Y}$$

$$\frac{\text{All } X \text{ (don't) see all } Z \quad \text{Some } Y \text{ are } Z}{\text{All } X \text{ (don't) see some } Y}$$

$$\frac{\text{All } Z \text{ (don't) see all } Y \quad \text{Some } X \text{ are } Z}{\text{Some } X \text{ (don't) see all } Y}$$

$$\frac{\text{Some } X \text{ don't see some } Y \quad \text{All } X \text{ see all } Y}{\text{No } X \text{ are } X}$$

$$\frac{\text{Some } X \text{ (don't) see some } Y}{\text{Some } Y \text{ is a } Y}$$

A complete (but infinite) system (for the positive fragment)

$$\frac{\textit{All } X^{\downarrow} \textit{ see all } Y^{\downarrow}}{\textit{All } X^{\downarrow} \textit{ see some } Y^{\uparrow}} \qquad \frac{\textit{Some } X^{\uparrow} \textit{ see all } Y^{\downarrow}}{\textit{Some } X^{\uparrow} \textit{ see some } Y^{\uparrow}}$$

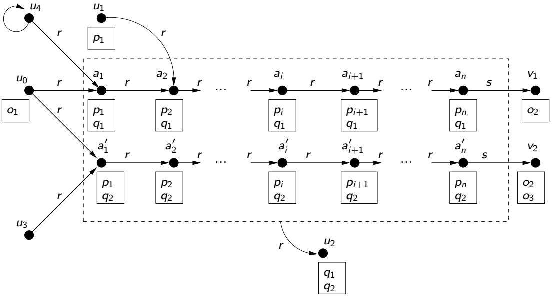
$$\frac{\textit{All X see NP} \quad \textit{Some X are Y}}{\textit{Some Y see NP}} \qquad \frac{\textit{Some X see NP}}{\textit{Some X is an X}}$$

$$\frac{\begin{matrix}\textit{All X see some } W_1 \\ \textit{All } W_1 \textit{ see some } W_2 \\ \vdots \\ \textit{All } W_n \textit{ see some Z}\end{matrix} \quad \textit{All X see all Z} \quad \textit{All Z are Y}}{\textit{All X see some Y}}$$

$$\frac{\begin{matrix}\textit{All Y see some } W_1 \\ \textit{All } W_1 \textit{ see some } W_2 \\ \vdots \\ \textit{All } W_n \textit{ see some Z}\end{matrix} \quad \textit{All Z see all Y} \quad \textit{All Z are Z} \quad \exists X}{\textit{Some X see all Y}}$$

More negative results

At this point, we know that $\mathcal{R}$ has no pure syllogistic proof system.
But it has an indirect system (one using RAA).
With a lot more work, one can show that $\mathcal{R}^{\dagger}$
doesn't even have an indirect system!


u4
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an
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pi+1 q1
pn q1
o2
s
a'1
a'2
a'i
a'i+1
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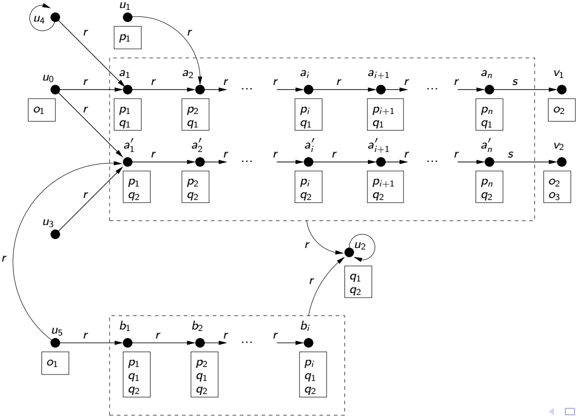
p2 q2
pi q2
pi+1 q2
pn q2
o2 o3
u3
u2
q1 q2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pi q2
pi+1 q2
pn q2
o2 o3
u3
u2
q1 q2
u5
o1
b1
b2
bi
p1 q1 q2
p2 q1 q2
pi q1 q2

Systems with relative clauses: $\mathcal{R}^*$ and $\mathcal{R}^{*\dagger}$

We still cannot cover

$$\frac{\textit{Every man is an animal}}{\textit{He who kills a man kills an animal}}$$

And so we must enrich the syntax further.
Recall that $\mathcal{R}^\dagger$ has sentences

$$Q_1\ X\ ((don't)\ see\ Q_2\ Y)$$

So one natural extension is to allow complex noun-expressions as subjects.
We'd get sentences like

$$All\ (see\ some\ X)\ (don't\ see\ all\ Y)$$

which we read as

$$All\ who\ see\ some\ X\ don't\ see\ all\ Y$$

This is the fragment $\mathcal{R}^{*\dagger}$.

Since $\mathcal{R}^{\star\dagger}$ includes $\mathcal{R}^{\dagger}$,
it's not so surprising that it admits no (indirect) syllogistic logic.
To sensibly restrict it, we recall $\mathcal{S}$ and $\mathcal{S}^{\dagger}$.
We take $\mathcal{R}^*$ to be the sentences of the forms

$$Q_1\ X\ ((\textit{don't})\ V\ Q_2\ Y)$$
$$Q_1\ (V_1\ Q_2\ Y)\ ((\textit{don't})\ V_2\ Q_2\ Y)$$

That is, the subject is either a non-negated noun X
or else a set expression *see some/all* X with a non-negated verb.
We also allow logically equivalent sentences.

An indirect system $\mathbf{R}^*$ for $\mathcal{R}^*$

$$\frac{\textit{All } X \textit{ are } Y}{\textit{All (see all } Y\textit{) (see all } X\textit{)}}$$

$$\frac{\textit{All } X \textit{ are } Y}{\textit{All (see some } X\textit{) (see some } Y\textit{)}}$$

$$\frac{\textit{Some } X \textit{ are } Y}{\textit{All (see all } X\textit{) (see some } Y\textit{)}}$$

$$\frac{\textit{Some } X \textit{ see some } Y}{\exists Y}$$

$$\frac{\textit{All } X \textit{ are } X'}{\textit{All (see all } Y\textit{) see all } X}$$

RAA

A result and an open problem

Theorem

If $P \neq NP$, then there are no finite sound and refutation-complete systems for $\mathcal{R}^*$.

Open problem

Eliminate the hypothesis $P \neq NP$.

Iterated relative clauses

$$\frac{\textit{All skunks are mammals}}{\textit{All who fear all who respect all skunks fear all who respect all mammals}}$$

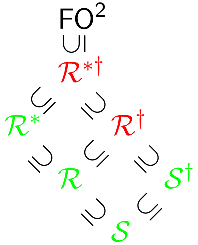
We cannot directly say this big sentence in this logic.
But we may add new X and Y, and then have

$$\Gamma \vdash \textit{All (fear all } X\textit{) (fear all } Y\textit{)}$$

where Γ is

All skunks are mammals
All X (respect all skunks)
All (respect all skunks) X
All Y (respect all mammals)
All (respect all mammals) Y

The Aristotle boundary



$\mathcal{S}$	direct, complete	NLOGSPACE
$\mathcal{S}^{\dagger}$	direct, complete	NLOGSPACE
$\mathcal{R}$	direct, refutation complete	NLOGSPACE
$\mathcal{R}^{\dagger}$	not even indirect	EXPTIME
$\mathcal{R}^{*}$	indirect, complete	Co-NPTIME [McA-G]
$\mathcal{R}^{*\dagger}$	not even indirect	EXPTIME
FO^2		NEXPTIME [GKV]

$\mathcal{S}$	NLOGSPACE	lower bound via reachability problem for directed graphs
$\mathcal{S}^{\dagger}$	NLOGSPACE	upper bound via 2SAT
$\mathcal{R}$	NLOGSPACE	upper bound takes special work
$\mathcal{R}^{\dagger}$	EXPTIME	lower bound via K^U
$\mathcal{R}^{*\dagger}$	EXPTIME	upper bound by Pratt-Hartmann 2004
$\mathcal{R}^{*}$	Co-NPTIME	essentially due to McAllester and Givan 1992
FO^2	NEXPTIME	Grädel, Kolaitis, and Vardi 1997

All xenophobics hate all actors
All yodelers hate all zookeepers
All non-yodelers hate all non-actors
All wardens are xenophobics

?? All wardens hate all zookeepers

1		*All xenophobics hate all actors*	Hyp
2		*All yodelers hate all zookeepers*	Hyp
3		*All non-yodelers hate all non-actors*	Hyp
4		*All wardens are xenophobics*	Hyp
5	*Jane*	*Jane is a warden*	Hyp
6		*All wardens are xenophobics*	R, 4
7		*Jane is a xenophobic*	All Eliim, 6
8		*All xenophobics hate all actors*	R, 2
9		*Jane hates all actors*	All Elim, 8
10		*Jane is a yodeler*	Hyp
11		*Jane hates all zookeepers*	Easy from 2
12		*Jane is not a yodeler*	Hyp
13		*Jane hates all zookeepers*	See below
14		*Jane hates all zookeepers*	Cases 10-11, 12-13
15		*All wardens hate all zookeepers*	All Intro

1	*Jane is not a yodeler*	Hyp
2	*Jane hates all actors*	R, above
3	*All non-yodelers hate all non-actors*	R, above
4	*Jane hates all non-actors*	All Elim, 1, 3
5	*Bob* | *Bob is a zookeeper*	Hyp
6	*Bob is actor*	Hyp
7	*Jane hates Bob*	All Elim, 2
8	*Bob is not actor*	Hyp
9	*Jane hates Bob*	All Elim, 4
10	*Jane hates Bob*	Cases
11	*Jane hates all zookeepers*	All Intro

A modal logic connection: transitive frames

This would be useful to axiomatize the logic of *comparative adjectives* such as is taller than, and also the *inside-of* and *isa* relations.

$$\frac{\textit{All } X \textit{ are taller than all } Y \quad \textit{Some } Y \textit{ are taller than some } Z}{\textit{All } X \textit{ are taller than some } Z}$$

Let's abbreviate this rule as

$$\frac{\begin{array}{c}\forall\forall \\ \exists\exists\end{array}}{\forall\exists}$$

Axiomatization on top of positive-$\mathcal{R}$: Sam Ziegler, August 2008

$$
\begin{array}{c}\forall\forall\\ \forall\forall\\ \hline \forall\forall\end{array}
\quad
\begin{array}{c}\forall\forall\\ \forall\exists\\ \hline \forall\exists\end{array}
\quad
\begin{array}{c}\forall\forall\\ \exists\forall\\ \hline \forall\forall\end{array}
\quad
\begin{array}{c}\forall\forall\\ \exists\exists\\ \hline \forall\exists\end{array}
$$

$$
\begin{array}{c}\forall\exists\\ \forall\forall\\ \hline \forall\forall\end{array}
\quad
\begin{array}{c}\forall\exists\\ \forall\exists\\ \hline \forall\exists\end{array}
\quad
\begin{array}{c}\exists\forall\\ \forall\forall\\ \hline \exists\forall\end{array}
\quad
\begin{array}{c}\exists\forall\\ \forall\exists\\ \hline \exists\exists\end{array}
$$

$$
\begin{array}{c}\exists\forall\\ \exists\forall\\ \hline \exists\forall\end{array}
\quad
\begin{array}{c}\exists\forall\\ \exists\exists\\ \hline \exists\exists\end{array}
\quad
\begin{array}{c}\exists\exists\\ \forall\forall\\ \hline \exists\forall\end{array}
\quad
\begin{array}{c}\exists\exists\\ \forall\exists\\ \hline \exists\exists\end{array}
$$

Schubert's Steamroller

Every wolf is a animal. Every fox is a animal.
Every bird is a animal. Every caterpillar is a animal.
Every snail is a animal. Some wolf exists.
Some fox exists. Some bird exists. Some caterpillar exists.
Some snail exists. Every grain is a plant. Some grain exists.
Every caterpillar is smaller than every bird.
Every snail is smaller than every bird.
Every bird is smaller than every fox.
Every fox is smaller than every wolf.
It is not true that some wolf eats some fox.
It is not true that some wolf eats some grain.
Every bird eats every caterpillar.
It is not true that some bird eats some snail. Every caterpillar eats some plant. Every snail eats some plant.
Every animal eats every plant or every animal that is smaller than itself and eats some plant.
Show that Every animal eats some animal that eats some grain.

What I hope to have explained

Extended syllogistic logics have

- ⋆ Ancient roots yet contemporary applications.
- ⋆ A plethora of different logical systems.
- ⋆ Connections to algebraic semantics.
- ⋆ Interesting proof theory.
- ⋆ Decidability/complexity results.