An Algebraic Generalization of Kripke structures

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Unital Involutive Quantale

A unital involutive quantale Q is a complete lattice equipped with an additional structure of involutive monoid,

- $(ab)c = a(bc)$
- $ae = a = ea$
- $a^{**} = a$
- $(ab)^* = b^*a^*$,

which is compatible with arbitrary joins:

- $(\bigvee_i a_i)b = \bigvee_i a_i b$
- $b(\bigvee_i a_i) = \bigvee_i b a_i$
- $(\bigvee_i a_i)^* = \bigvee_i a_i^*$.

In other words, an involutive monoid in the monoidal category of sup-lattices.(Notation: $1 = \bigvee Q$ and $0 = \bigvee \emptyset$)

Example - Unital Involutive Quantale of Relations

$2^{W \times W}$ is a unital involutive quantale:

- Multiplication of binary relations is given by (forward) composition: $R.S = R; S = S \circ R$.
- The multiplicative unit e is the identity (or diagonal) relation $\Delta_W = \{(x, x) \mid x \in W\}$.
- The involution is reversal: $R^* = \{(y, x) \mid xRy\}$.

$W = \{0, 1, 2\}$

$R = \{(0, 1), (1, 1)\}$, $S = \{(1, 2), (1, 0), (2, 1)\}$
$R.S = \{(0, 2), (0, 0), (1, 2), (1, 0)\}$
$\Delta_W = \{(0, 0), (1, 1), (2, 2)\}$
$R^* = \{(1, 0), (1, 1)\}$
Remark: if $R, S \subseteq \Delta_W$ then $R.S = R \cap S$

Stably supported quantale

Let Q be a unital involutive quantale. A stable support on Q is a join preserving map

$$\varsigma : Q \to Q$$

satisfying, for all $a, b \in Q$,

$$\varsigma a \leq e \quad (1)$$
$$\varsigma a \leq aa^* \quad (2)$$
$$a \leq \varsigma a a \quad (3)$$
$$\varsigma(ab) \leq \varsigma a \quad (4)$$

A stably supported quantale (ssq) is a unital involutive quantale equipped with a stable support.

Remarks about SSQ's

1. $\downarrow e$ is a frame, for $a, b \in \downarrow e$ we have $ab = a \wedge b$ and $a^* = a = \varsigma a.(\downarrow e = \varsigma Q)$
2. In a ssq, $\varsigma a = e \wedge aa^* = e \wedge a1$.
3. Being stably supported is a property of a unital involutive quantale, rather than additional structure.
4. The homomorphisms of unital involutive quantales necessarily preserve the support.
5. The category of ssqs is a full reflective subcategory of the category of unital involutive quantales.

Example - Ssq of Binary Relations

Let $R \in 2^{W \times W}$, we define:

$$\mathrm{dom}\, R = \{x \in W \mid xRy \text{ for some } y \in W\} .$$

Using the identification $W \cong \Delta_W$

$$x \mapsto (x, x)$$

we may equivalently define it to be the (subdiagonal) relation

$$\varsigma R = \{(x, x) \in W \times W \mid x \in \mathrm{dom}\, R\} ,$$

thus turning dom into an operation

$$\varsigma : 2^{W \times W} \to 2^{W \times W}$$

It is easy to check that ς satisfies the conditions of a stably support.

Formulas - Basic type modal language

$$\varphi ::= \text{propositional symbol} \mid \neg\varphi \mid \varphi \wedge \psi \mid \Diamond\varphi$$

Kripke models

A Kripke Model is a triple (W, R, V) where:

- W is the set of worlds
- $R \subseteq W \times W$ is the accessibility relation
- V : Formulas $\rightarrow 2^W$ is the valuation map, which satisfies

$$
\begin{aligned}
V(\varphi \wedge \psi) &= V(\varphi) \cap V(\psi) \\
V(\neg \varphi) &= W \setminus V(\varphi) \\
V(\Diamond \varphi) &= \{x \in W \mid xRy \text{ for some } y \in V(\varphi)\}
\end{aligned}
$$

Kripke models - shifting to the quantale language

A Kripke model can be equivalently defined to be a triple (W, R, V), where the valuation map

$$V : \text{Formulas} \to 2^{\Delta_W} (\cong 2^W)$$

satisfies

$$\begin{aligned} V(\varphi \wedge \psi) &= V(\varphi); V(\psi) \\ V(\neg\varphi); V(\varphi) &= \emptyset \\ V(\neg\varphi) \cup V(\varphi) &= \Delta_W \\ V(\Diamond\varphi) &= \varsigma(R; V(\varphi)) \end{aligned}$$

The properties of V are entirely defined in terms of the structure of unital involutive quantale of $2^{W \times W}$ together with its unique support ς.

Abstract Kripke Model

An abstract Kripke model of the basic type language of propositional modal logic is a triple (Q, r, v) consisting of

- an ssq Q
- an accessibility element $r \in Q$
- a valuation map $v : \text{Formulas} \to \varsigma Q$

satisfying:

$$
\begin{aligned}
v(\varphi \wedge \psi) &= v(\varphi)v(\psi) & [= v(\varphi) \wedge v(\psi)] \\
v(\neg\varphi)v(\varphi) &= 0 \\
v(\neg\varphi) \vee v(\varphi) &= e \\
v(\Diamond\varphi) &= \varsigma(r\, v(\varphi))
\end{aligned}
$$

We interpret the formulas inside a Boolean subalgebra of ςQ.
To get an intuitionistic version, we replace the two middle conditions by a single one using the pseudo-complement in ςQ (this is a frame and therefore a Heyting algebra): $v(\neg\varphi) = v(\varphi) \to 0$

K,T,S4,S5

An abstract Kripke model without any restriction on the accessibility element is a K-model.
Particular cases are easily captured:

- T-models: The accessibility element r satisfies $e \leq r$ ("reflexivity").
- S4-models: The accessibility element satisfies $e \leq r = r^2$ ("reflexivity" and "transitivity").
- S5-models: The accessibility element satisfies $e \leq r = r^2 = r^*$ ("reflexivity", "transitivity" and "symmetry").

Additional examples

In this setting the logic of programs PDL, and the ramified temporal logic CTL are easily dealt with, in the paper you will find a quantale semantics for these logics. Also, the notion of metric spaces has been generalized to metric quantales in a way yielding an immediate application of this kind of semantics to the study of modal logic of metric spaces.

PDL - Language

$\alpha ::=$ atomic programs $\mid \alpha;\beta \mid \alpha^* \mid \alpha \cup \beta \mid \varphi?$

(here φ is a formula) and each program determines a modality:

$\varphi ::=$ atomic formulas $\mid \neg\varphi \mid \varphi \wedge \psi \mid \langle\alpha\rangle\varphi$.

- Atomic programs – indecomposable (execute in single step)
- $\alpha;\beta$ – "Do α, then β"
- $\alpha \cup \beta$ – nondeterministic choice between running α or β
- α^* – a finite number of executions of α
- $\varphi?$ – evaluates φ at the current state, continuing if and only if it is true.

PDL - Models

A PDL-model is a triple $(\mathrm{Q}, \pi, \mathrm{v})$ where Q is an ssq and π and v are maps

$$\pi : \text{Programs} \to \mathrm{Q}$$
$$\mathrm{v} : \text{Formulas} \to \varsigma\mathrm{Q}$$

that satisfy the conditions:

$$\begin{aligned}
\pi(\alpha;\beta) &= \pi(\alpha)\pi(\beta) \\
\pi(\alpha^*) &= \bigvee_{\mathrm{n}\in\mathbb{N}} \pi(\alpha)^{\mathrm{n}} \\
\pi(\alpha \cup \beta) &= \pi(\alpha) \vee \pi(\beta) \\
\pi(\varphi?) &= \mathrm{v}(\varphi) \\
\mathrm{v}(\langle\alpha\rangle\varphi) &= \varsigma(\pi(\alpha)\,\mathrm{v}(\varphi)) \\
\text{etc.} &
\end{aligned}$$

We have defined a new interpretation of modal logic, extending the classic one (based on Kripke models).

Each system is still sound w.r.t. the correspondent abstract models.

We are just adding more models.

From the classic completeness results we get immediately completeness, but this is not the whole story.

Let us see why in the algebraic setting we are working on.

Lindembaum Quantale for system K

Let $\mathfrak{B}_K$ be lindenbaum algebra for K.
There is an ssq $\mathfrak{Q}_K$ presented by generators and relations, using $\mathfrak{B}_K \cup \{r\}$ with $r \notin \mathfrak{B}_K$ as set of generators and imposing:

$$\begin{aligned} [x \vee y] &= [x] \vee [y] \\ [\neg x][x] &= 0 \\ [\neg x] \vee [x] &= e \\ [\Diamond x] &= \varsigma([r][x]) \, . \end{aligned}$$

Assuring the preservation of $\mathfrak{B}_K$'s modal algebra structure.

The same applies to: T, S4, S5, propositional dynamic logic, etc., defining the appropriate "Lindenbaum quantales".
E.g. $\mathfrak{Q}_{S5}$ is obtained as before, imposing in addition the relations:

$$e \leq r = r^2 = r^* \, .$$

Recovering the classical models

The Lindenbaum quantale has a universal property analogous to that of the Lindenbaum algebra:
there is a bijective correspondence between abstract Kripke models (Q, r, v) and homomorphims of unital involutive quantales $\mathfrak{Q} \longrightarrow Q$.

If W is a set then a homomorphism

$$\rho : \mathfrak{Q} \longrightarrow 2^{W \times W}$$

is the same as a Kripke model with set of possible worlds W and accessibility relation $\rho(\alpha)$ where α is the accessibility element of $\mathfrak{Q}$.

Remark

With modal algebras the modal operators $\Diamond$ (or $\langle\alpha\rangle$, etc.) have to be specified in advance and have to be preserved by the homomorphisms, here the algebra of unital involutive quantales is common to any of the modal logics we have seen so far.

Classical completeness

Completeness corresponds to

$$\eta : \mathfrak{B} \to \mathfrak{Q}$$

being injective.

This is implied by the classical completeness theorem since the composition with the morphism correspondent to the canonical model is injective:

$$\mathfrak{B} \to \mathfrak{Q} \to 2^{\mathrm{W} \times \mathrm{W}} .$$

However the axiom of choice is being used to construct the set of maximal consistent sets of formulas W.

It is natural to try to prove it algebraically in a more direct way, in particular one that will be valid in an arbitrary topos: such a proof of injectivity is what we mean by constructive completeness (still open for K, T, K4, and S4, but S5 is already known to be constructively complete).

Bimodal frames

Let (Q, α) be a pointed ssq. Then the frame ςQ is canonically equipped with the two unary sup-latice endomorphisms $\lozenge$ and $\blacklozenge$ defined by, for each $x \in \varsigma Q$,

$$\begin{aligned} \lozenge x &= \varsigma(\alpha x) \\ \blacklozenge x &= \varsigma(\alpha^* x) , \end{aligned}$$

which are easily seen to satisfy the following conjugacy conditions:

$$\begin{aligned} \lozenge x \wedge y &\leq \lozenge(x \wedge \blacklozenge y) \\ \blacklozenge x \wedge y &\leq \blacklozenge(x \wedge \lozenge y) . \end{aligned}$$

Such a structure $(L, \lozenge, \blacklozenge)$, where L is a frame and $\lozenge$ and $\blacklozenge$ satisfy the conjugacy conditions, will be called a bimodal frame.

Taking the support yields a functor from the category of pointed ssq to the category of bimodal frames.

From modal algebras to bimodal frames

Given a modal algebra B we call $B^{\blacklozenge}$ to the bimodal algebra obtained by adjoining $\blacklozenge$ and imposing the conjugacy conditions.

From a bimodal algebra with $\lozenge$ and $\blacklozenge$ satisfying the conjugacy conditions we get a bimodal frame by ideal completion.

To each system (T, K4, S4, S5) corresponds a class of bimodal frames satisfying the extra axioms.
E.g. T-bimodal frames satisfy $\lozenge\lozenge\varphi \leq \lozenge\varphi$ and $\blacklozenge\blacklozenge\varphi \leq \blacklozenge\varphi$

Bimodal Frames and Pointed Ssq

For each of the systems K, T, K4, S4, and S5, we obtain adjunctions between the correspondent categories of bimodal frames and pointed ssqs, all of them being co-reflections.

Sketch on how to obtain a pointed ssq from a bimodal frame

- From $(L, \Diamond, \blacklozenge)$ we define unital involutive "tensor quantale"

  $$\mathcal{T}(L) = \bigoplus_{I} L^{(w)}$$

  I is the free involutive monoid in one generator α and $L^{(w)} = L^{\otimes(|w|+1)}$.
  The product and the involution are defined in the pure tensors:
  $(x_0 \otimes \ldots \otimes x_n)(y_0 \otimes \ldots \otimes y_m) = x_0 \otimes \ldots \otimes x_n \wedge y_0 \otimes y_1 \otimes \ldots \otimes y_m$
  $(x_1 \otimes \cdots \otimes x_n)^* = x_n \otimes \cdots \otimes x_1$
  ς is defined inductively on pure tensors,
  - $\varsigma x = x$, if $x \in L^{(\varepsilon)}$;
  - $\varsigma x = x_0 \wedge \langle w_1 \rangle(\varsigma x')$, if $n \geq 1$.

  where $x' = x_1 \otimes \cdots \otimes x_n \in L^{(w_2 \cdots w_n)}$ and
  $\langle w_i \rangle$ is $\Diamond$ or $\blacklozenge$ according to whether $w_i = \alpha$ or $w_i = \alpha^*$, respectively.
- We obtain a stably supported quotient of $\mathcal{T}(L)$, $\mathfrak{T}_K(L)$ (this is the left adjoint of taking the support).
- Again, if we impose the correspondent conditions on the accessibility element we will have $\mathfrak{T}_T(L), \mathfrak{T}_{S4}(L) \ldots$.

Constructive Completeness

As a result, in the particular case of system K, we get:

- the injection $\mathrm{Idl}(\mathfrak{B}_K^{\blacklozenge}) \xrightarrow{1-1} \mathfrak{T}_K(\mathrm{Idl}(\mathfrak{B}_K^{\blacklozenge}))$
- $\mathfrak{T}_K(\mathfrak{B}_K^{\blacklozenge})$ has the universal property of $\mathfrak{Q}_K$.

and since $\mathfrak{B}_K^{\blacklozenge} \xrightarrow{1-1} \mathrm{Idl}(\mathfrak{B}_K^{\blacklozenge})$ all of this sums up to:

$$\mathfrak{B}_K \xrightarrow{1-1?} \mathfrak{B}_K^{\blacklozenge} \xrightarrow{1-1} \mathfrak{T}_K(\mathrm{Idl}(\mathfrak{B}_K^{\blacklozenge})) \cong \mathfrak{Q}_K$$

Axiomatization of S5: S4 + $(\Diamond = \blacklozenge)$ as a corollary

For S5, $\mathfrak{B}_{S5} \to \mathfrak{B}^{\blacklozenge}_{S5}$ is trivially 1-1.
S5 is complete for the following axiom schemata:

$$
\begin{array}{rcl}
\varphi & \to & \Diamond\varphi \\
\Diamond\Diamond\varphi & \to & \Diamond\varphi \\
\Diamond\varphi \wedge \psi & \to & \Diamond(\varphi \wedge \Diamond\psi) \;\; \text{(instead of } \varphi \to \Box\Diamond\varphi)
\end{array}
$$

No use of negation or the modal necessity operator $\Rightarrow$ intuitionistically useful.

New possibilities

There are plenty of examples of ssqs besides the quantales of binary relations arising from various geometric or analytic structures.

Thus we are provided with a uniform way of defining semantic interpretations of propositional modal logic based on such structures.

Example

Let G be a groupoid (a small category all of whose arrows have are isomorphisms), we write G_0 for the set of objects and G_1 the set of arrows. 2^{G_1} is a stably supported quantale:

$$\begin{aligned} UV &= \{xy \mid x \in U,\ y \in V,\ r(x) = d(y)\} \\ e &= \{Id_G : G \in G_0\} \\ U^* &= \{x^{-1} : x \in U\} \end{aligned}$$

The powerset of a discrete group, and the quantale of binary relations on a set is a particular case of this.

More generally, the topology $\Omega(G)$ of any topological étale groupoid G is a sub-ssq of 2^G.

Other Examples

Some examples of groupoids:

- The fundamental groupoid of a topological space.
- The monodromy groupoid of a foliation (a generalization of the previous example).
- The holonomy groupoid of a foliation.
- The dual groupoid of a C*-algebra.

Groupoids can be constructed from arbitrary inverse semigroups.
Some examples of inverse semigroups are:

- The partial bijections on a set X (the symmetric inverse semigroup of X).
- The locally defined homeomorphisms of a topological space (pseudo-group).
- The locally defined diffeomorphisms of a smooth manifold.
- Any semigroup of partial isometries on a Hilbert space, or, more generally, of a C*-algebra.

So...

If we replace $2^{W \times W}$ by a more general quantale like $\Omega(G)$, hence taking as models of propositional modal logic the homomorphisms
$\mathfrak{Q} \to \Omega(G)$
instead of
$\mathfrak{Q} \to 2^{W \times W}$ (where $\mathfrak{Q}$ is a Lindenbaum quantale), we are led in a natural way to semantics which may be interesting, for example, for dealing with hybrid systems.

Conclusion

Quantale semantics automatically provides a bridge between modal logic and those areas of mathematics where examples of étale groupoids and inverse semigroups occur, such as operator algebras and differential topology.

BIBLIOGRAPHY

The notion of supported quantale and its relation to inverse semigroups and groupoids can be found in
P. Resende, Étale groupoids and their quantales,
which can be downloaded from http://arxiv.org/abs/math/0412478.

The connection between (pseudo)metric spaces and metric quantales has been addressed in the diploma thesis of Ana Toledo that will come out as a paper soon .

The applications to modal logic are covered in
S. Marcelino and P. Resende, Algebraic Generalization of Kripke structures,
which can be downloaded from http://arxiv.org/abs/0704.1886.