RESTRICTED INTERPOLATION IN MODAL AND SUPERINTUITIONISTIC LOGICS

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ABSTRACT

Restricted interpolation property IPR is investigated. It is proved that in finite slice extensions of the Grzegorczyk logic and in finite slice superintuitionistic logics IPR is equivalent to the projective Beth property PB2.

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INTRODUCTION

Restricted interpolation property IPR in modal logics was introduced in [10, 11] in connection with studying projective Beth properties. On the class of normal modal logics (n.m.l.) IPR follows from the deductive interpolation property IPD, and also from the projective Beth property PB2. On the contrary, neither IPD nor PB2 follow from IPR on the class of modal logics [15]. Note that in modal logics over S5 IPR is equivalent to the Craig interpolation property CIP, and also to IPD and PB2 [15].

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There is a close connection between different versions of interpolation and Beth properties in logics and amalgamation and surjectivity of epimorphisms in corresponding varieties of algebras [2, 4, 6, 15, 16]. In particular, the projective Beth property PB2 is equivalent to the strong epimorphisms surjectivity, and the restricted interpolation property IPR to the restricted amalgamation property of the corresponding variety [11].

It is known [6] that only finitely many normal extensions of S4 have CIP or IPD. In [7, 9, 12] we investigated projective Beth properties in non-classical logics and their relations with interpolation properties. In [7] an exhaustive list of superintuitionistic logics (s.i.l.) with PB2 was found. This list turned out to be finite. The logics with PB2 are fully described, and decidability of PB2 on the class of superintuitionistic calculi is proved [8]. Analogous results for extensions of the Grzegorczyk logic Grz are obtained in [14], and for positive logics in [9].

But up to now we do not know if the number of s.i.l. with IPR is finite. The same problem is unsolved for n.m.l. over K4, Grz and S4.

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It is proved in [9] that in positive logics, as well as in s.i.l. and in n.m.l., PB2 follows from CIP and implies IPR. It is proved in [13] that IPR and PB2 are equivalent in positive logics and also in logics over s.i.l. KC and over Grz.2.

In the present paper we prove that IPR and PB2 are equivalent in finite slice extensions of the intuitionistic logic Int and of Grz. Moreover, we prove the equivalence of IPR and PB2 in all modal logics containing Grz and not contained in $\Delta(\mathrm{Grz.2})$. In addition, IPR and PB2 are equivalent in all s.i.l. over $\Delta(\mathrm{KC})$. An example of n.m.l. with IPD but without the Beth property B2 [6] shows that IPR is not equivalent to PB2 in modal logics. But the problems of equivalence of IPR and PB2 in s.i.l. and in modal logics over Grz, S4, K4 remain open.

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NTERPOLATION AND BETH PROPERTIES

If $\mathbf{p}$ is a list of propositional variables, let $A(\mathbf{p})$ denote a formula whose all variables are in $\mathbf{p}$, and $\mathcal{F}(\mathbf{p})$ the set of all such formulas.

Let L be a logic, $\vdash_L$ a consequence relation associated with L. Suppose that $\mathbf{p}$, $\mathbf{q}$, $\mathbf{r}$ are disjoint lists of propositional variables. We consider the languages which contain at least one constant $\top$ ("truth") or $\bot$ ("false"). One can define two *interpolation properties CIP* and *IPD* as follows:

CIP. If $\vdash_L A(\mathbf{p},\mathbf{q}) \to B(\mathbf{p},\mathbf{r})$, then there exists a formula $C(\mathbf{p})$ such that $\vdash_L A(\mathbf{p},\mathbf{q}) \to C(\mathbf{p})$ and $\vdash_L C\mathbf{p}) \to B(\mathbf{p},\mathbf{r})$.

IPD. If $A(\mathbf{p},\mathbf{q}) \vdash_L B(\mathbf{p},\mathbf{r})$, then there exists a formula $C(\mathbf{p})$ such that $A(\mathbf{p},\mathbf{q}) \vdash_L C(\mathbf{p})$ and $C\mathbf{p}) \vdash_L B(\mathbf{p},\mathbf{r})$.

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In [11] a *restricted interpolation property* was introduced:
IPR. If $A(\mathbf{p},\mathbf{q}), B(\mathbf{p},\mathbf{r}) \vdash_L C(\mathbf{p})$, then there exists a formula $A'(\mathbf{p})$ such that $A(\mathbf{p},\mathbf{q}) \vdash_L A'(\mathbf{p})$ and $A'(\mathbf{p}), B(\mathbf{p},\mathbf{r}) \vdash_L C(\mathbf{p})$.

Most known modal logics such as Lewis' systems S4 and S5, Grzegorczyk's logic Grz, the logic GL of provability, the logic K4 and the least normal modal logic K have CIP.

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The Beth definability properties

have as their source the theorem on implicit definability proved by E.Beth in 1953 for the classical first order logic: Any predicate implicitly definable in a first order theory is explicitly definable. We define two versions PB1 and PB2 of *projective Beth property*.

Let $\mathbf{x}$, $\mathbf{q}$, $\mathbf{q}'$ be disjoint lists of variables not containing y and z.
PB1. If $\vdash_L A(\mathbf{x}, \mathbf{q}, y) \& A(\mathbf{x}, \mathbf{q}', z) \to (y \leftrightarrow z)$, then $\vdash_L A(\mathbf{x}, \mathbf{q}, y) \to (y \leftrightarrow B(\mathbf{x}))$ for some formula $B(\mathbf{x})$.
PB2. If $A(\mathbf{x}, \mathbf{q}, y), A(\mathbf{x}, \mathbf{q}', z) \vdash_L (y \leftrightarrow z)$, then $A(\mathbf{x}, \mathbf{q}, y) \vdash_L (y \leftrightarrow B(\mathbf{x}))$ for some formula $B(\mathbf{x})$.
We get weaker versions B1 and B2 of Beth's property by deleting $\mathbf{q}, \mathbf{q}'$ in PB1 and PB2 respectively.

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- In the family NE(K) of normal modal logics:
  PB1 $\Longleftrightarrow$ B1 $\Longleftrightarrow$ CIP $\Rightarrow$ IPD $\Rightarrow$ IPR, PB1 $\Rightarrow$ PB2 $\Rightarrow$ IPR+B2,
  the pairs B2 and IPD, PB2 and IPD, B2 and IPR are incomparable.
- In NE(K4): CIP $\Rightarrow$ IPD $\Rightarrow$ PB2 $\Rightarrow$ IPR.
- In NE(S4): CIP $\Rightarrow$ IPD $\Rightarrow$ PB2 $\Rightarrow$ IPR, IPD $\not\Rightarrow$ CIP.
- In NE(Grz): CIP $\Leftrightarrow$ IPD $\Rightarrow$ PB2 $\Rightarrow$ IPR, PB2 $\not\Rightarrow$ IPD.
- In NE(S5): CIP $\Leftrightarrow$ IPD $\Leftrightarrow$ PB2 $\Leftrightarrow$ IPR.

All logics over Int and over Int^+ satisfy B1 and B2, and for these logics we have:

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PROJECTIVE BETH PROPERTY OVER INT AND GRZ

In this section we recall the results obtained in previous works. It is well known that there is a translation T from Int into S4. With any logic M in $NE(\mathrm{S4})$ its *intuitionistic fragment* $\rho(M) = \{A \mid T(A) \in M\}$ is associated; M is said to be a *modal companion of* $\rho(M)$. Moreover, ρ is an isomorphism between $NE(\mathrm{Grz})$ and the family $E(\mathrm{Int})$ of s.i.l. (see [6]).

In [7] all superintuitionistic logics with the projective Beth property PB2 are described. It turned out that there are exactly 16 logics possessing PB2, and all of them are finitely axiomatizable and finitely approximable. In [14] we proved that there are exactly 13 logics with PB2 in NE(Grz).

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Recall that
$\mathrm{Grz} = \mathrm{S4} + (\Box(\Box(p \to \Box p) \to p) \to p)$; $\mathrm{Grz.2} = \mathrm{Grz} + \mathrm{S4.2}$;
for $L \in NE(\mathrm{S4})$:

$$\Delta(L) = \mathrm{S4} + \{\Box(p \to \Box A) \vee \Box(\neg p \to \Box A) | A \in L \text{ and } p \text{ is not in } A\}.$$

Denote

$$\sigma_0 = \bot,\ \sigma_{n+1} = \Box p_{n+1} \vee \Box(\Box p_{n+1} \to \sigma_n).$$

A logic $L \in NE(\mathrm{S4})$ is said to be a *logic of the slice* $\mathcal{S}_n$ if $L \vdash \sigma_n$ and $L \nvdash \sigma_{n-1}$; L is a logic *of finite slice* if it is a logic of some slice $\mathcal{S}_n$ $(n \geq 0)$, and a logic *of infinite slice* otherwise.
It is known (see, for instance, [6]), that a logic in $NE(\mathrm{S4})$ is locally tabular if and only if it is a logic of finite slice.

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A logic $L \in NE(\mathrm{S4})$ is said to be a *logic of the slice* $\mathcal{S}_n$ if $L \vdash \sigma_n$ and $L \nvdash \sigma_{n-1}$; L is a logic *of finite slice* if it is a logic of some slice $\mathcal{S}_n$ $(n \geq 0)$, and a logic *of infinite slice* otherwise.
It is known (see, for instance, [6]), that a logic in $NE(\mathrm{S4})$ is locally tabular if and only if it is a logic of finite slice.

This classification is closely connected with classification of superintuitionistic logics introduced by T. Hosoi. For any $L \in NE(\mathrm{S}4)$, the modal logic L is a logic of the slice $\mathcal{S}_n$ if and only if $\rho(L)$ is a superintuitionistic logic of the n-th slice.

Theorem

For any logic L of finite slice in NE(Grz), L has PB2 iff $\rho(L)$ has PB2, and L has IPR iff $\rho(L)$ has IPR.

Theorem

[12, 14] *In NE(Grz) there are exactly 13 logics with PB2, among them there are exactly ten finite slice logics.*

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Consider the logics of infinite slice. There are three extensions of the Grzegorczyk logic, which do not possess PB2, but their intuitionistic fragments have PB2.

We recall some denotations of superintuitionistic logics:

$$\mathrm{KC} = \mathrm{Int} + (\neg p \vee \neg\neg p),$$

$$\mathrm{LC} = \mathrm{Int} + (p \to q) \vee (q \to p),$$

$$\Delta(L) = \mathrm{Int}{+}\{p{\vee}(p \to A) \mid A \in L \text{ and } p \text{ has no occurrences in } A\}.$$

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Theorem

[7, 12] *Superintuitionistic logics* LC, $\Delta(\mathrm{LC})$, *and* $\Delta(\mathrm{LC}) + \mathrm{KC}$ *have PB2 but no modal companion of these logics has PB2 or even IPR.*

We note that the logics Grz and Grz.2 have CIP (Boolos 1980, Rautenberg 1983) and, consequently, PB2. These logics are modal companions of the intermediate logics Int and KC respectively. Also the logic $\Delta(\mathrm{Grz.2})$, which is the greatest modal companion of $\Delta(\mathrm{KC})$, has PB2.

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MAIN RESLULTS

Recall [7, 14] that any logic with IPR in $E(\mathrm{Int})$ or in $NE(\mathrm{Grz})$ is a logic of infinite slice or of a finite slice of a number $n \leq 3$.

Theorem

Let L be a superintuitionistic logic or a modal logic in $NE(\mathrm{Grz})$, and L a logic of finite slice. Then L has IPR if and only if L has PB2.

The proof is rather complicated and contains a lot of technical details. We apply algebraic semantics and the equivalence of IPR and PB2 to the restricted amalgamation property and strong epimorphisms surjectivity, respectively (see [11]).

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As for logics of infinite slice, we can extend this theorem. Recall [13] that IPR and PB2 are equivalent over Grz.2. We can generalise this result and Theorem 4 as follows.

Theorem

Let $L \in NE(\mathrm{Grz})$, $L \not\subseteq \Delta(\mathrm{Grz.2})$. Then L has IPR if and only if L has PB2.

Thus the problem of restricted interpolation over Grz is open only for logics contained in $\Delta(\mathrm{Grz.2})$.

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For superintuitionistic logics we proved the equivalence of IPR and PB2 over KC [13]. We extend this result as follows.

Theorem

For any logic in $E(\Delta(\mathrm{KC}))$, IPR is equivalent to PB2.

So the problem remains open for infinite slice s.i.l. not containing $\Delta(\mathrm{KC})$.

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Algebraic characterization of superintuitionistic logics is built with the use of Heyting algebras, and characterization of logics in $NE(\mathrm{Grz})$ with the use of Grzegorczyk algebras.

If $\mathbf{A}$ is a Heyting algebra, we denote by $\mathbf{A}^+$ the Heyting algebra obtained from $\mathbf{A}$ by adding a new greatest element.

Theorem

Let s.i.l. L have IPR and satisfies the condition: for any finite Heyting algebra $\mathbf{A}$,

$$\mathbf{A}^+ \in V(L) \Rightarrow \mathbf{A}^{++} \in V(L). \quad (1)$$

Then L coincides with one of the logics LC, KC *or* Int, *and so has CIP.*

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The condition (1) of Proposition 7 is not necessary. We formulate a necessary and sufficient condition.

Theorem

A superintuitionistic logic L with IPR has PB2 if and only if it satisfies the condition: for any Heyting algebra $\mathbf{A}$ *and any finite Heyting algebra* $\mathbf{B}^+$*, which is a homomorphic image of* $\mathbf{A}$*,*

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