A Modal Perspective on MSO Alternation Hierarchies

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9 September 2008
Introduction

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- Quantifiers range over sets of possible worlds.
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Quantifiers range over sets of possible worlds.

$SOPML$ admits a prenex normal form representation (ten Cate, 2005).

Is the quantifier alternation hierarchy infinite?
We show that the quantifier alternation hierarchy is infinite over finite directed graphs.

This holds for both *pointed models* (i.e., local perspective) and *models/frames* (i.e., global perspective).

As a by-product, we show that

\[ SOPML + \text{Universal Modality} = \text{MSO}. \]
Our Plan:

1. We identify classes of directed graphs where $SOPML = MSO$. 
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2. We show that the \( MSO \) alternation hierarchy is infinite over these classes.
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1. We identify classes of directed graphs where $SOPML = MSO$.

2. We show that the $MSO$ alternation hierarchy is infinite over these classes.

3. We conclude that the $SOPML$ alternation hierarchy is infinite over the classes.
Preliminaries

- \( \text{VAR} = \text{VAR}_{FO} \cup \text{VAR}_{SO} \)  
  \((\text{first-order and second-order variable symbols})\)

- \( \text{PROP} = \{p_x \mid x \in \text{VAR}_{FO}\} \cup \{p_X \mid X \in \text{VAR}_{SO}\} \)  
  \((\text{proposition variables})\)

- A model is a model of predicate logic. A pointed model is a pair \((M, w)\), where \(M\) is a model and \(w \in \text{Dom}(M)\).

- We reserve \( \models \) for modal and \( \models \) for predicate logic.
Preliminaries

▶ Syntactic alternation hierarchies:

\[
\Sigma_n^{MSO} = \{ \exists X \forall Y \exists Z \ldots n \text{ blocks } (\varphi) \mid \varphi \in FO \}
\]

\[
\Sigma_n^{SOPML} = \{ \exists p_x \forall p_y \exists p_z \ldots n \text{ blocks } (\varphi) \mid \varphi \in ML \}
\]

▶ Semantic alternation hierarchies:

\[
\Sigma_n^{MSO} = \{ \text{Mod}(\varphi) \mid \varphi \in \Sigma_n^{MSO} \}
\]

\[
\Sigma_n^{SOPML} = \{ \text{Mod}(\varphi) \mid \varphi \in \Sigma_n^{SOPML} \}
\]
Language $L(\tau)$ of SOPML:

\[
\begin{align*}
P_i \in \tau & \Rightarrow p_i \in L(\tau) \\
p_x \in PROP & \Rightarrow p_x \in L(\tau) \\
R_j \in \tau, \ \varphi \in L(\tau) & \Rightarrow \diamond_j \varphi \in L(\tau) \\
\varphi \in L(\tau) & \Rightarrow \neg \varphi \in L(\tau) \\
\varphi_1, \varphi_2 \in L(\tau) & \Rightarrow \varphi_1 \wedge \varphi_2 \in L(\tau) \\
\varphi \in L(\tau), \ p_x \in PROP & \Rightarrow \exists p_x \varphi \in L(\tau)
\end{align*}
\]
Semantics:

\( (M, w), V \models p_i \iff w \in P_i^M \)
\( (M, w), V \models p_x \iff w \in V(p_x) \)
\( (M, w), V \models \neg \varphi \iff (M, w), V \not\models \varphi \)
\( (M, w), V \models \varphi \land \psi \iff (M, w), V \models \varphi \) and \( (M, w), V \models \psi \)
\( (M, w), V \models \exists p_x \varphi \iff \exists U \subseteq \text{Dom}(M)(\text{Dom}(M), V[p_x \mapsto U] \models \varphi) \)
\( (M, w), V \models \Box_j \varphi \iff \exists u \in \text{Dom}(M)(wR_j u \text{ and } (M, u) \models \varphi) \)

Valuation function \( V : \text{PROP} \rightarrow \mathcal{P}(	ext{Dom}(M)) \) interprets proposition variables in \( \text{PROP} \).
Second-Order Propositional Modal Logic (SOPML)

- **SOPML** on models:
  \[
  M \models \varphi \iff \forall w \in \text{Dom}(M)((M, w) \models \varphi)
  \]

- **MSO** on pointed models:
  \[
  (M, w) \models \psi(x) \iff M, f[x \mapsto w] \models \psi(x)
  \]
Language $L^E(\tau)$ of $SOPML(E)$:

$$\varphi \in L(S) \Rightarrow \varphi \in L^E(S)$$

$$\varphi \in L^E(S) \Rightarrow \langle E \rangle \varphi \in L^E(S)$$

Truth definition:

$$(M, w), V \models \langle E \rangle \varphi \iff \exists u \in Dom(M)((M, u), V \models \varphi)$$
SOPML(E) = MSO

MSO translates to SOPML(E):

\[ TR(P_i(x)) = \langle E \rangle (p_i \land p_x) \]
\[ TR(X(y)) = \langle E \rangle (p_x \land p_y) \]
\[ TR(R_j(x,y)) = \langle E \rangle (p_x \land \Diamond_j p_y) \]
\[ TR(x = y) = \langle E \rangle (p_x \land p_y) \]
\[ TR(\neg \psi) = \neg TR(\psi) \]
\[ TR(\psi \land \varphi) = TR(\psi) \land TR(\varphi) \]
\[ TR(\exists x \, \psi) = \exists p_x (uniq(p_x) \land TR(\psi)) \]
\[ TR(\exists X \, \psi) = \exists p_X (TR(\psi)) \]

\[ uniq(p_x) = \langle E \rangle p_x \land \forall p_y (\langle E \rangle (p_y \land p_x) \rightarrow \Box (p_x \rightarrow p_y)) \]
Theorem
$\text{SOPML}(E) = \text{MSO}$ over pointed models.

Proof.
Associate assignment function
$f : \text{VAR} \rightarrow \text{Dom}(M) \cup \mathcal{P}(\text{Dom}(M))$ with a related valuation
function $V_f : \text{PROP} \rightarrow \mathcal{P}(\text{Dom}(M))$ such that

1. $V_f(p_x) = \{f(x)\}$ for all $x \in \text{VAR}_{FO}$
2. $V_f(p_X) = f(X)$ for all $X \in \text{VAR}_{SO}$.

Prove that $(M, w), f \models \psi(x) \iff (M, w), V_f \models TR(\psi(x))$.  □
Theorem

\(\text{SOPML}(E) = \text{MSO over models.}\)

Proof.
If not trivial, then straightforward.
Definition
Structure \( S = (W, R, ...) \) (where \( R \subseteq W \times W \)) is \textit{localized} if there exists a point \( w \in W \) such that \( wRu \) for all \( u \in W \). Point \( w \) is a \textit{localizer}.

Definition
If \((M, w)\) is a pointed model where \( w \) is a localizer, we say that \((M, w)\) is \textit{l-pointed}.
SOPML = MSO on Pointed Localized Models

\[
\begin{align*}
LTR(P_i(x)) &= \Diamond(p_i \land p_x) \\
LTR(X(y)) &= \Diamond(p_x \land p_y) \\
LTR(R_j(x, y)) &= \Diamond(p_x \land \Diamond j \ p_y) \\
LTR(x = y) &= \Diamond(p_x \land p_y) \\
LTR(\neg \psi) &= \neg TR(\psi) \\
LTR(\psi \land \varphi) &= TR(\psi) \land TR(\varphi) \\
LTR(\exists x \ \psi) &= \exists p_x(uniq'(p_x) \land TR(\psi)) \\
LTR(\exists X \ \psi) &= \exists p_x(TR(\psi)) \\
uniq'(p_x) &= \Diamond p_x \land \forall p_y (\Diamond(p_y \land p_x) \rightarrow \Box(p_x \rightarrow p_y))
\end{align*}
\]
Lemma

$\text{SOPML} = \text{MSO on l-pointed models.}$

Proof.

Similar to the proof of the assertion that $\text{MSO} = \text{SOPML}(E)$ on pointed models.
Let $C$ be a class of localized models. Let $\varphi$ be an SOPML-sentence such that for each model $M \in C$,

1. There exists at least one point $w \in \text{Dom}(M)$ that satisfies $\varphi$.
2. Every point $w \in \text{Dom}(M)$ that satisfies $\varphi$, is a localizer.

We say that $\varphi$ \textit{fixes localizers} on $C$. 

Lemma

If some SOPML-sentence fixes localizers on class $C$ of models, then $MSO = SOPML$ on $C$. 
Summing up:

- $\text{SOPML} = \text{MSO}$ on 1-pointed models.

- If some $\text{SOPML}$-sentence fixes localizers on class $C$ of models, then $\text{SOPML} = \text{MSO}$ on $C$. 
Our plan:

1. We identify classes of directed graphs where $SOPML = MSO$.

2. We show that the $MSO$ alternation hierarchy is infinite over these classes.

3. We conclude that the $SOPML$ alternation hierarchy is infinite over the classes.
MSO over Grids

MSO alternation hierarchy over GRIDS is strict:

Theorem (Schweikardt 1997)

For all $n \in \mathbb{N}_{\geq 1}$, $\Sigma_n^{MSO}(GRID) \neq \Sigma_{n+1}^{MSO}(GRID)$. 
Figure: An example of a grid.
Encode grids as localized directed graphs:

Figure: Each point of a grid is replaced by a two-point gadget.
Figure: A grid and its encoding. The localizer connects to each point of the graph; for the sake of clarity, most arrows originating from the localizer have not been drawn.
Let $\alpha : GRID \rightarrow GRAPH$ denote the encoding map.

$\text{Dom}(\alpha(Gd)) = \{(x, 0), (x, 1) \mid x \in \text{Dom}(Gd)\}$, where $(x, 0)$ is the reflexive point and $(x, 1)$ the irreflexive one.
Lemma

Encoding $\alpha : \text{GRID} \rightarrow \text{GRAPH}$ is injective in the following sense: If $\alpha(Gd) \cong \alpha(Gd')$, then $Gd \cong Gd'$. □
Lemma
For every grid-sentence $\varphi_{GRID} \in \Sigma_n$ there exists a graph-sentence $\varphi_{GRAPH} \in \Sigma_n$ such that

$$Gd \models \varphi_{GRID} \iff \alpha(Gd) \models \varphi_{GRAPH}.$$ 

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Proof.

Given variable assignment $f_{GRID}$, define $f_{GRAPH}:

$$f_{GRAPH}(y) = (f_{GRID}(y), 0)$$

$$f_{GRAPH}(Y) = f_{GRID}(Y) \times \{0\}$$

Now prove

$$Gd, f_{GRID} \models \varphi_{GRID} \iff \alpha(Gd), f_{GRAPH} \models \varphi_{GRAPH}.$$
Obtaining $\varphi_{\text{GRAPH}}$ from $\varphi_{\text{GRID}}$

Now, for example, if $\varphi_{\text{GRID}} = xS_1y$, then $\varphi_{\text{GRAPH}}$ is

$$
\begin{align*}
\psi_t(x) \land \psi_t(y) & \rightarrow \bot \\
\land \psi_t(x) \land \neg \psi_t(y) & \rightarrow \forall z(z Ry \rightarrow (\psi_t(z) \lor z = y)) \\
\land \neg \psi_t(x) \land \psi_t(y) & \rightarrow \bot \\
\land \neg \psi_t(x) \land \neg \psi_t(y) & \rightarrow xRy \land x \neq y
\end{align*}
$$

where $\psi_t(x)$ states that $x$ is the localizer of the graph.
Assume (induction hypothesis) that
\( Gd, f_{GRID} \models \pi_{GRID} \iff \alpha(Gd), f_{GRAPH} \models \pi_{GRAPH}. \)

- If \( \varphi_{GRID} = \exists x(\pi_{GRID}) \), then \( \varphi_{GRAPH} = \exists x(xRx \land \pi_{GRAPH}). \)

- If \( \varphi_{GRID} = \exists X(\pi_{GRID}) \), then
  \( \varphi_{GRAPH} = \exists X(\forall x(X(x) \rightarrow xRx) \land \pi_{GRAPH}). \)
Lemma

For every graph-sentence $\varphi_{\text{GRAPH}} \in \Sigma_n$, there is a grid-sentence $\varphi_{\text{GRID}} \in \Sigma_n$ such that

$$Gd \models \varphi_{\text{GRID}} \iff \alpha(Gd) \models \varphi_{\text{GRAPH}}.$$ 

Proof.

Somewhat technical...
 Obtaining $\varphi_{GRID}$ from $\varphi_{GRAPH}$

Partition the graph domain into four sets:

1. $V_{t_0} = \{localizer\}$
2. $V_{t_1} = \{t_1\}$ (see the picture)
3. $V_0 = \{x \mid xRx\} \setminus \{localizer\}$
4. $V_1 = \{x \mid \neg xRx\} \setminus \{t_1\}$
Obtaining $\varphi_{GRID}$ from $\varphi_{GRAPH}$

We then define assignment $f_{GRID}$ from $f_{GRAPH}$.

First, define

$$VAR_{GRID} = VAR_{FO} \cup \{ X^{t_0}, X^{t_1}, X^1, X^0 \mid X \in VAR_{SO} \}.$$
Obtaining $\varphi_{GRID}$ from $\varphi_{GRAPH}$

\[
\begin{align*}
  f_{GRID}(X^0) &= \{ u \in \text{Dom}(Gd) \mid (u, 0) \in f_{GRAPH}(X) \cap V_0 \} \\
  f_{GRID}(X^1) &= \{ u \in \text{Dom}(Gd) \mid (u, 1) \in f_{GRAPH}(X) \cap V_1 \} \\
  f_{GRID}(X^{t_0}) &= \{ u \in \text{Dom}(Gd) \mid (u, 0) \in f_{GRAPH}(X) \cap V_{t_0} \} \\
  f_{GRID}(X^{t_1}) &= \{ u \in \text{Dom}(Gd) \mid (u, 1) \in f_{GRAPH}(X) \cap V_{t_1} \}
\end{align*}
\]
Obtaining $\varphi_{GRID}$ from $\varphi_{GRAPH}$

For first-order variables, we let

$$f_{GRID}(x) = u \iff \left( f_{GRAPH}(x) = (u, 0) \text{ or } f_{GRAPH}(x) = (u, 1) \right)$$
Unfortunately, we cannot prove the following:

For every graph-formula $\varphi_{GRAPH} \in \Sigma_n$, there exists a grid-formula $\varphi_{GRID} \in \Sigma_n$ such that

$$Gd, f_{GRID} \models \varphi_{GRID} \iff \alpha(Gd), f_{GRAPH} \models \varphi_{GRAPH}.$$
Any function $\kappa : \text{VAR}_{FO} \rightarrow \{ t_0, t_1, 0, 1 \}$ is an assignment type.

Assignment $f_{\text{GRAPH}}$ is of type $\kappa$ if $f_{\text{GRAPH}}(x) \in V_{\kappa}(x)$ for all $x \in \text{VAR}_{FO}$.
Obtaining $\varphi_{GRID}$ from $\varphi_{GRAPH}$

We then prove the following:

For every assignment type $\kappa$ and every graph-formula $\varphi_{GRAPH} \in \Sigma_n$ there exists a grid-formula $\varphi_{GRID}^{\kappa} \in \Sigma_n$ such that

$$Gd, f_{GRID} \models \varphi_{GRID}^{\kappa} \iff \alpha(Gd), f_{GRAPH} \models \varphi_{GRAPH}$$

holds for all $f_{GRAPH}$ of type $\kappa$. $\square$
Call grid encodings localized grid graphs.

Proposition

MSO alternation hierarchy is strict over localized grid graphs.
Our plan:

1. We identify classes of directed graphs where \( SOPML = MSO \).

2. We show that the \( MSO \) alternation hierarchy is infinite over these classes.

3. We conclude that the \( SOPML \) alternation hierarchy is infinite over the classes.
Theorem

SOPML alternation hierarchy over pointed finite directed graphs is infinite.

Proof.

We define the class of \( l \)-pointed localized grid graphs (identify localizers as evaluation points). We know that \( MSO = SOPML \) on \( l \)-pointed structures.
Theorem

**SOPML alternation hierarchy over finite directed graphs is infinite.**

Proof.

Formula

\[
\forall p_x (p_x \rightarrow \Diamond p_x)
\]

\[
\land
\]

\[
\forall p_x (p_x \rightarrow \exists p_y (\neg p_y \land \Diamond (p_y \land \Diamond p_x)))
\]

fixes localizers on localized grid graphs. Thus **SOPML = MSO** over localized grid graphs.

\[\square\]
Corollary

SOPML alternation hierarchy is infinite over Kripke frames.