A Modal Perspective on MSO Alternation Hierarchies

Antti Kuusisto

9 September 2008

Introduction

- $SOPML \;=\; ML + \;\exists p, \forall p$

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- Quantifiers range over sets of possible worlds.

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- Quantifiers range over sets of possible worlds.
- *SOPML* admits a prenex normal form representation (ten Cate, 2005).

Introduction

Open question, (van Benthem [Modal Logic and Classical Logic, 1983] and ten Cate [Expressivity of Second-Order Propositional Modal Logic, Journal of Phil. Log., 2006]):

Is the quantifier alternation hierarchy infinite?

Introduction

- We show that the quantifier alternation hierarchy is infinite over finite directed graphs.
- This holds for both *pointed models* (i.e., local perspective) and *models/frames* (i.e., global perspective).
- As a by-product, we show that

$$SOPML + \ Universal\ Modality = MSO.$$

Our Plan:

1. We identify classes of directed graphs where $SOPML = MSO$.

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2. We show that the MSO alternation hierarchy is infinite over these classes.

3. We conclude that the $SOPML$ alternation hierarchy is infinite over the classes.

Preliminaries

- $VAR \; = \; VAR_{FO} \cup VAR_{SO}$
  (*first-order and second-order variable symbols*)

- $PROP \; = \; \{p_x \mid x \in VAR_{FO}\} \cup \{p_X \mid X \in VAR_{SO}\}$
  (*proposition variables*)

- A model is a model of predicate logic. A pointed model is a pair (M, w), where M is a model and $w \in Dom(M)$.

- We reserve $\Vdash$ for modal and $\models$ for predicate logic.

Preliminaries

- Syntactic alternation hierarchies:

$$\Sigma_n^{MSO} = \{\ \overline{\exists X}\ \overline{\forall Y}\ \overline{\exists Z}\ ...n\ blocks\ (\varphi) \mid \varphi \in FO\}$$

$$\Sigma_n^{SOPML} = \{\ \overline{\exists p_x}\ \overline{\forall p_y}\ \overline{\exists p_z}\ ...n\ blocks\ (\varphi) \mid \varphi \in ML\}$$

- Semantic alternation hierarchies:

$$\underline{\Sigma_n^{MSO}} = \{\ Mod(\varphi) \mid \varphi \in \Sigma_n^{MSO}\}$$

$$\underline{\Sigma_n^{SOPML}} = \{\ Mod(\varphi) \mid \varphi \in \Sigma_n^{SOPML}\}$$

Language $L(\tau)$ of *SOPML*:

$$
\begin{array}{lcl}
P_i \in \tau & \Rightarrow & p_i \in L(\tau) \\
p_x \in PROP & \Rightarrow & p_x \in L(\tau) \\
R_j \in \tau, \quad \varphi \in L(\tau) & \Rightarrow & \Diamond_j \, \varphi \in L(\tau) \\
\varphi \in L(\tau) & \Rightarrow & \neg\varphi \in L(\tau) \\
\varphi_1, \varphi_2 \in L(\tau) & \Rightarrow & \varphi_1 \wedge \varphi_2 \in L(\tau) \\
\varphi \in L(\tau), \ p_x \in PROP & \Rightarrow & \exists p_x \, \varphi \ \in L(\tau)
\end{array}
$$

Second-Order Propositional Modal Logic (SOPML)

Semantics:

$$
\begin{array}{lcl}
(M,w),V \Vdash p_i & \Leftrightarrow & w \in P_i^M \\
(M,w),V \Vdash p_x & \Leftrightarrow & w \in V(p_x) \\
(M,w),V \Vdash \neg\varphi & \Leftrightarrow & (M,w),V \nVdash \varphi \\
(M,w),V \Vdash \varphi \wedge \psi & \Leftrightarrow & (M,w),V \Vdash \varphi \text{ and } (M,w),V \Vdash \psi \\
(M,w),V \Vdash \exists p_x\, \varphi & \Leftrightarrow & \exists U \subseteq Dom(M)((M,w),V[p_x \mapsto U] \Vdash \varphi) \\
(M,w),V \Vdash \Diamond_j\, \varphi & \Leftrightarrow & \exists u \in Dom(M)(wR_j u \text{ and } (M,u) \Vdash \varphi)
\end{array}
$$

Valuation function $V : PROP \longrightarrow \mathcal{P}(Dom(M))$ interprets proposition variables in $PROP$.

Second-Order Propositional Modal Logic (SOPML)

- *SOPML* on models:

$$M \Vdash \varphi \;\Leftrightarrow\; \forall w \in Dom(M)((M, w) \Vdash \varphi)$$

- *MSO* on pointed models:

$$(M, w) \models \psi(x) \;\Leftrightarrow\; M, f[x \mapsto w] \models \psi(x)$$

SOPML with Universal Modality (SOPML(E))

Language $L^E(\tau)$ of $SOPML(E)$:

$$\begin{array}{lcl} \varphi \in L(S) & \Rightarrow & \varphi \in L^E(S) \\ \varphi \in L^E(S) & \Rightarrow & \langle E \rangle \varphi \in L^E(S) \end{array}$$

Truth definition:

$$(M, w), V \Vdash \langle E \rangle \varphi \ \Leftrightarrow \ \exists u \in Dom(M)((M, u), V \Vdash \varphi)$$

SOPML(E) = MSO

MSO translates to *SOPML(E)*:

$$\begin{array}{lcl}
TR(P_i(x)) & = & \langle E\rangle(p_i \wedge p_x) \\
TR(X(y)) & = & \langle E\rangle(p_X \wedge p_y) \\
TR(R_j(x,y)) & = & \langle E\rangle(p_x \wedge \Diamond_j\ p_y) \\
TR(x = y) & = & \langle E\rangle(p_x \wedge p_y) \\
TR(\neg\psi) & = & \neg TR(\psi) \\
TR(\psi \wedge \varphi) & = & TR(\psi) \wedge TR(\varphi) \\
TR(\exists x\ \psi) & = & \exists p_x(uniq(p_x) \wedge TR(\psi)) \\
TR(\exists X\ \psi) & = & \exists p_X(TR(\psi))
\end{array}$$

$$uniq(p_x) = \langle E\rangle p_x\ \wedge\ \forall p_y(\ \langle E\rangle(p_y \wedge p_x)\ \rightarrow\ [E](p_x \rightarrow p_y)\ )$$

SOPML(E) = MSO

Theorem

$SOPML(E) = MSO$ over pointed models.

Proof.

Associate assignment function $f : VAR \longrightarrow Dom(M) \cup \mathcal{P}(Dom(M))$ with a related valuation function $V_f : PROP \longrightarrow \mathcal{P}(Dom(M))$ such that

1. $V_f(p_x) = \{f(x)\}$ for all $x \in VAR_{FO}$
2. $V_f(p_X) = f(X)$ for all $X \in VAR_{SO}$.

Prove that $(M, w), f \models \psi(x) \Leftrightarrow (M, w), V_f \Vdash TR(\psi(x))$. □

SOPML(E) = MSO

Theorem
SOPML(E) = MSO over models.

Proof.
If not trivial, then straightforward. □

Localized Structures

Definition

Structure $S = (W, R, ...)$ (where $R \subseteq W \times W$) is *localized* if there exists a point $w \in W$ such that wRu for all $u \in W$. Point w is a *localizer.*

Definition

If (M, w) is a pointed model where w is a localizer, we say that (M, w) is *l-pointed.*

SOPML = MSO on Pointed Localized Models

$$
\begin{array}{lcl}
LTR(P_i(x)) & = & \Diamond(p_i \wedge p_x) \\
LTR(X(y)) & = & \Diamond(p_X \wedge p_y) \\
LTR(R_j(x,y)) & = & \Diamond(p_x \wedge \Diamond_j\ p_y) \\
LTR(x = y) & = & \Diamond(p_x \wedge p_y) \\
LTR(\neg\psi) & = & \neg TR(\psi) \\
LTR(\psi \wedge \varphi) & = & TR(\psi) \wedge TR(\varphi) \\
LTR(\exists x\ \psi) & = & \exists p_x(uniq'(p_x) \wedge TR(\psi)) \\
LTR(\exists X\ \psi) & = & \exists p_X(TR(\psi))
\end{array}
$$

$$
uniq'(p_x) = \Diamond p_x \ \wedge\ \forall p_y(\ \Diamond(p_y \wedge p_x)\ \rightarrow\ \Box(p_x \rightarrow p_y)\ )
$$

SOPML = MSO on l-pointed Models

Lemma

SOPML = MSO on l-pointed models.

Proof.

Similar to the proof of the assertion that $MSO = SOPML(E)$ on pointed models. □

SOPML on Models

Let C be a class of localized models. Let φ be an *SOPML*-sentence such that for each model $M \in C$,

1. There exists at least one point $w \in Dom(M)$ that satisfies φ.
2. Every point $w \in Dom(M)$ that satisfies φ, is a localizer.

We say that φ *fixes localizers* on C.

SOPML on Models

Lemma

If some SOPML-sentence fixes localizers on class C of models, then $MSO = SOPML$ on C.

□

Summing up:

- *SOPML* = *MSO* on *l*-pointed models.
- If some *SOPML*-sentence fixes localizers on class C of models, then *SOPML* = *MSO* on C.

Our plan:

1. We identify classes of directed graphs where $SOPML = MSO$.

2. We show that the MSO alternation hierarchy is infinite over these classes.

3. We conclude that the $SOPML$ alternation hierarchy is infinite over the classes.

MSO over Grids

MSO alternation hierarchy over *GRIDS* is strict:

Theorem (Schweikardt 1997)

For all $n \in \mathbb{N}_{\geq 1}$, $\underline{\Sigma_n^{MSO}}(GRID) \neq \underline{\Sigma_{n+1}^{MSO}}(GRID)$.

MSO over Grids

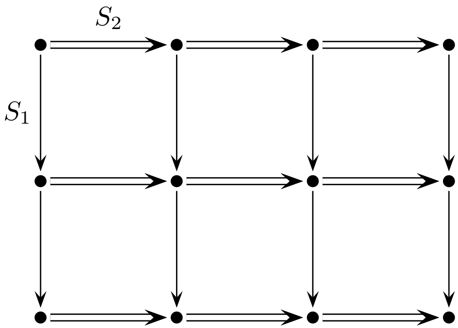


Figure: An example of a *grid*.

Encoding Grids as Localized Graphs

Encode grids as localized directed graphs:

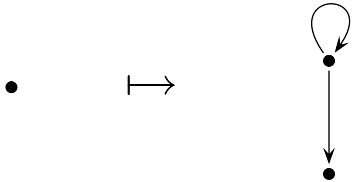


Figure: Each point of a grid is replaced by a two-point gadget.

Encoding Grids as Localized Graphs

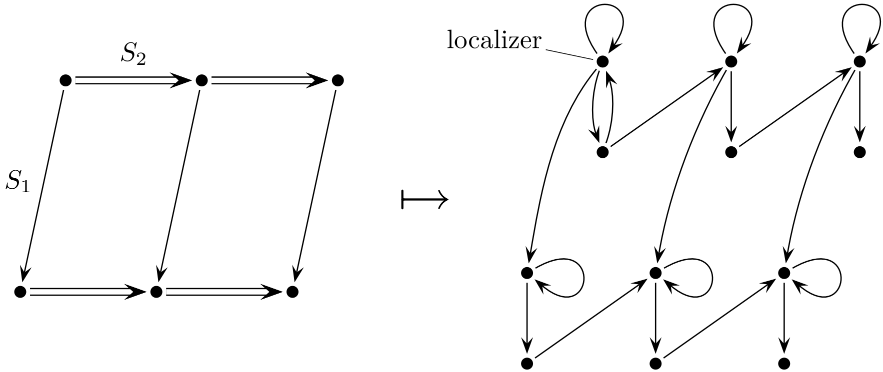


Figure: A grid and its encoding. The localizer connects to each point of the graph; for the sake of clarity, most arrows originating from the localizer have not been drawn.

Encoding Grids as Localized Graphs

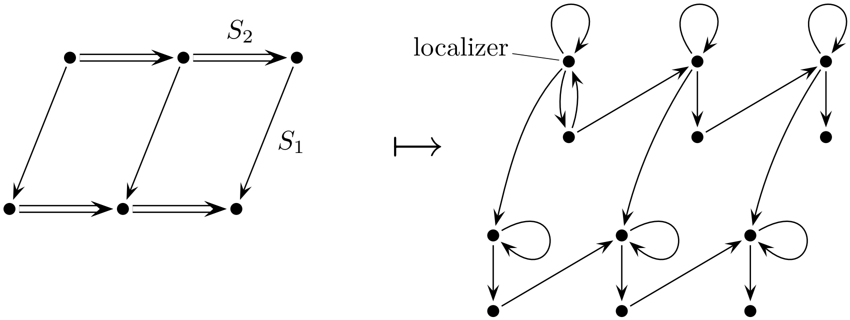
- Let $\alpha : GRID \longrightarrow GRAPH$ denote the encoding map.

- $Dom(\alpha(Gd)) = \{(x,0),\ (x,1) \mid x \in Dom(Gd)\}$,
  where $(x,0)$ is the reflexive point and $(x,1)$ the irreflexive one.

Encoding Grids as Localized Graphs

Lemma

Encoding $\alpha : GRID \longrightarrow GRAPH$ is injective in the following sense: If $\alpha(Gd) \cong \alpha(Gd')$, then $Gd \cong Gd'$. □



MSO over Grids vs. MSO over Grid Encodings

Lemma

For every grid-sentence $\varphi_{GRID} \in \Sigma_n$ there exists a graph-sentence $\varphi_{GRAPH} \in \Sigma_n$ such that

$$Gd \models \varphi_{GRID} \quad \Leftrightarrow \quad \alpha(Gd) \models \varphi_{GRAPH}.$$

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Obtaining φ_{GRAPH} from φ_{GRID}

Lemma

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$$Gd \models \varphi_{GRID} \quad \Leftrightarrow \quad \alpha(Gd) \models \varphi_{GRAPH}.$$

Proof.

Given variable assignment f_{GRID}, define f_{GRAPH}:

$$f_{GRAPH}(y) = (f_{GRID}(y),\ 0)$$

$$f_{GRAPH}(Y) = f_{GRID}(Y) \times \{0\}$$

Now prove

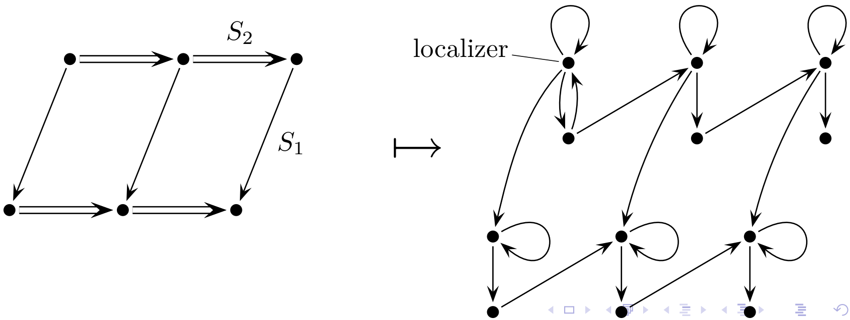
$$Gd, f_{GRID} \models \varphi_{GRID} \quad \Leftrightarrow \quad \alpha(Gd), f_{GRAPH} \models \varphi_{GRAPH}.$$

Obtaining φ_{GRAPH} from φ_{GRID}

Now, for example, if $\varphi_{GRID} = xS_1y$, then φ_{GRAPH} is

$$\begin{array}{rrcl} & \psi_{t_0}(x) \wedge \psi_{t_0}(y) & \rightarrow & \bot \\ \wedge & \psi_{t_0}(x) \wedge \neg\psi_{t_0}(y) & \rightarrow & \forall z(zRy \rightarrow (\psi_{t_0}(z) \vee z = y)) \\ \wedge & \neg\psi_{t_0}(x) \wedge \psi_{t_0}(y) & \rightarrow & \bot \\ \wedge & \neg\psi_{t_0}(x) \wedge \neg\psi_{t_0}(y) & \rightarrow & xRy \wedge x \neq y \end{array}$$

where $\psi_{t_0}(x)$ states that x is the localizer of the graph.



Obtaining φ_{GRAPH} from φ_{GRID}

Assume (induction hypothesis) that
$Gd, f_{GRID} \models \pi_{GRID} \quad \Leftrightarrow \quad \alpha(Gd), f_{GRAPH} \models \pi_{GRAPH}$.

- If $\varphi_{GRID} = \exists x(\pi_{GRID})$, then $\varphi_{GRAPH} = \exists x(xRx \wedge \pi_{GRAPH})$.
- If $\varphi_{GRID} = \exists X(\pi_{GRID})$, then
  $\varphi_{GRAPH} = \exists X(\forall x(X(x) \rightarrow xRx) \wedge \pi_{GRAPH})$. □

Obtaining φ_{GRID} from φ_{GRAPH}

Lemma

For every graph-sentence $\varphi_{GRAPH} \in \Sigma_n$, there is a grid-sentence $\varphi_{GRID} \in \Sigma_n$ such that

$$Gd \models \varphi_{GRID} \quad \Leftrightarrow \quad \alpha(Gd) \models \varphi_{GRAPH}.$$

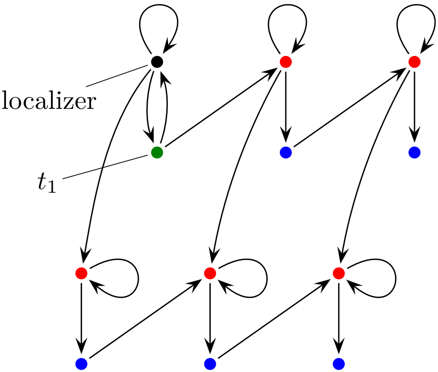
Proof.

Somewhat technical...

Obtaining φ_{GRID} from φ_{GRAPH}

Partition the graph domain into four sets:

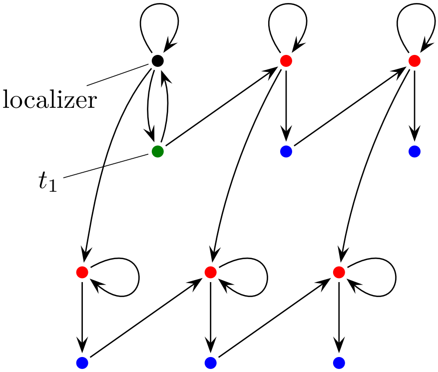
1. $V_{t_0} = \{localizer\}$
2. $V_{t_1} = \{t_1\}$ (*see the picture*)
3. $V_0 = \{x \mid xRx\} \setminus \{localizer\}$
4. $V_1 = \{x \mid \neg xRx\} \setminus \{t_1\}$



Obtaining φ_{GRID} from φ_{GRAPH}

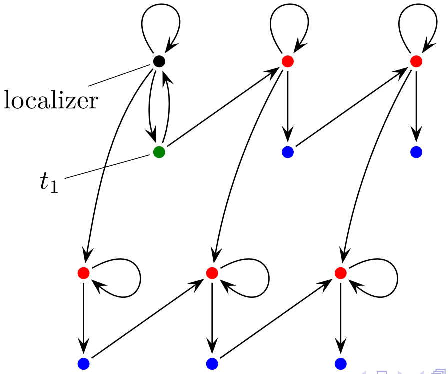
We then define assignment f_{GRID} from f_{GRAPH}.

First, define
$VAR_{GRID} = VAR_{FO} \cup \{\ X^{t_0},\ \ X^{t_1},\ \ X^1,\ \ X^0\ \mid X \in VAR_{SO}\ \}$.



Obtaining φ_{GRID} from φ_{GRAPH}

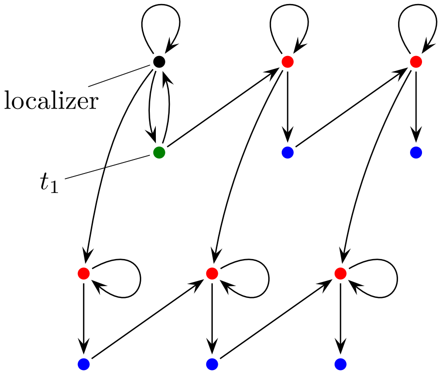
$$
\begin{aligned}
f_{GRID}(\ X^0\ ) &= \{u \in Dom(Gd) \mid (u,0) \in f_{GRAPH}(X) \cap V_0\ \} \\
f_{GRID}(\ X^1\ ) &= \{u \in Dom(Gd) \mid (u,1) \in f_{GRAPH}(X) \cap V_1\ \} \\
f_{GRID}(\ X^{t_0}\ ) &= \{u \in Dom(Gd) \mid (u,0) \in f_{GRAPH}(X) \cap V_{t_0}\ \} \\
f_{GRID}(\ X^{t_1}\ ) &= \{u \in Dom(Gd) \mid (u,1) \in f_{GRAPH}(X) \cap V_{t_1}\ \}
\end{aligned}
$$



Obtaining φ_{GRID} from φ_{GRAPH}

For first-order variables, we let

$$f_{GRID}(x) = u \quad \Leftrightarrow \quad \Big(f_{GRAPH}(x) = (u,0) \text{ or } f_{GRAPH}(x) = (u,1)\Big)$$



Obtaining φ_{GRID} from φ_{GRAPH}

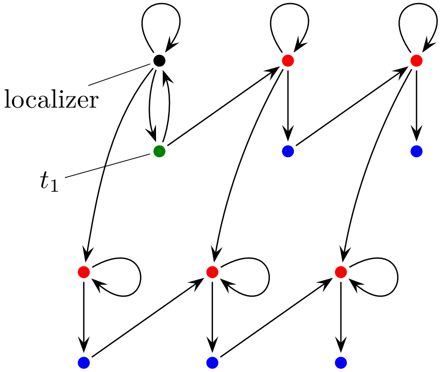
Unfortunately, we cannot prove the following:

For every graph-formula $\varphi_{GRAPH} \in \Sigma_n$, there exists a grid-formula $\varphi_{GRID} \in \Sigma_n$ such that

$$Gd, f_{GRID} \models \varphi_{GRID} \quad \Leftrightarrow \quad \alpha(Gd), f_{GRAPH} \models \varphi_{GRAPH}.$$

Obtaining φ_{GRID} from φ_{GRAPH}

Any function $\kappa : VAR_{FO} \longrightarrow \{\ t_0,\ \ t_1,\ \ 0,\ \ 1\ \}$ is an *assignment type*.



Assignment f_{GRAPH} is of type κ if $f_{GRAPH}(x) \in V_{\kappa(x)}$ for all $x \in VAR_{FO}$.

Obtaining φ_{GRID} from φ_{GRAPH}

We then prove the following:

For every assigment type κ and every graph-formula $\varphi_{GRAPH} \in \Sigma_n$ there exists a grid-formula $\varphi^{\kappa}_{GRID} \in \Sigma_n$ such that

$$Gd, f_{GRID} \models \varphi^{\kappa}_{GRID} \quad \Leftrightarrow \quad \alpha(Gd), f_{GRAPH} \models \varphi_{GRAPH}$$

holds for all f_{GRAPH} of type κ. □

MSO over Grid Encodings

Call grid encodings *localized grid graphs*.

Proposition

MSO alternation hierarchy is strict over localized grid graphs.

□

Our plan:

1. We identify classes of directed graphs where $SOPML = MSO$.

2. We show that the MSO alternation hierarchy is infinite over these classes.

3. We conclude that the $SOPML$ alternation hierarchy is infinite over the classes.

SOPML Hierarchy over Pointed Graphs

Theorem

SOPML alternation hierarchy over pointed finite directed graphs is infinite.

Proof.

We define the class of l-pointed localized grid graphs (identify localizers as evaluation points). We know that $MSO = SOPML$ on l-pointed structures. □

SOPML Hierarchy over Graphs

Theorem

SOPML alternation hierarchy over finite directed graphs is infinite.

Proof.

Formula

$$\begin{aligned} & \forall p_x(p_x \rightarrow \Diamond p_x) \\ \wedge & \\ & \forall p_x(p_x \rightarrow \exists p_y(\neg p_y \wedge \Diamond(p_y \wedge \Diamond p_x))) \end{aligned}$$

fixes localizers on localized grid graphs. Thus $SOPML = MSO$ over localized grid graphs. □

Corollary

SOPML alternation hierarchy is infinite over Kripke frames.

□