

Completeness for Moss's Coalgebraic Logic (Boolean version)

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coalgebras

coalgebra $X \rightarrow TX$

$T : \text{Set} \rightarrow \text{Set}$ *weak-pullback preserving* functor

Paradigmatic example: $T = \mathcal{P}$ (powerset-functor: coalgebras are Kripke frames)

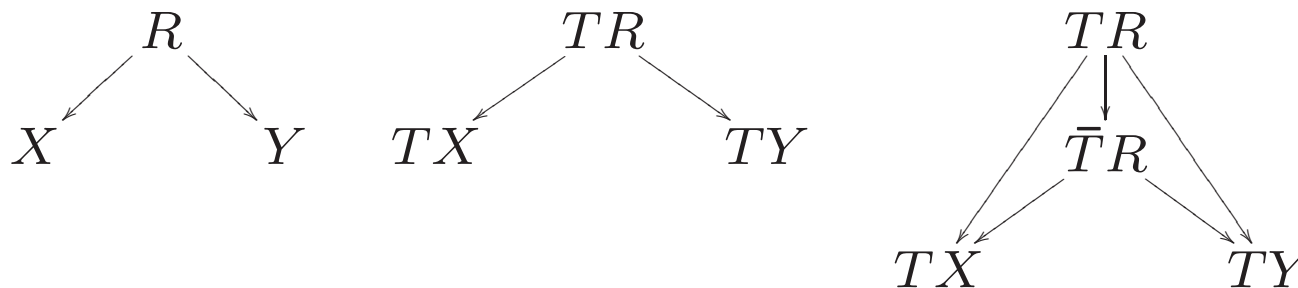
Other examples: Labelling of states and transitions (input and output), deterministic automata, probabilistic transition systems, stochastic transition systems, arbitrary combinations of these: infinitely many examples

Non-example: neighbourhood frames are coalgebras for a non weak-pullback-preserving functor

weak pullback preserving functors ...

... lift from Set to Rel (sets with relations as arrows),
ie, for each $T : \text{Set} \rightarrow \text{Set}$ we have $\bar{T} : \text{Rel} \rightarrow \text{Rel}$.

A relation is a span, to which we can apply T



\bar{T} is a functor iff T preserves weak pullbacks

Some obvious good properties hold ...

Moss's coalgebraic logic

Given T

The language \mathcal{L} is closed under Boolean operations and

if $\alpha \in T_\omega \mathcal{L}$ then $\nabla \alpha \in \mathcal{L}$

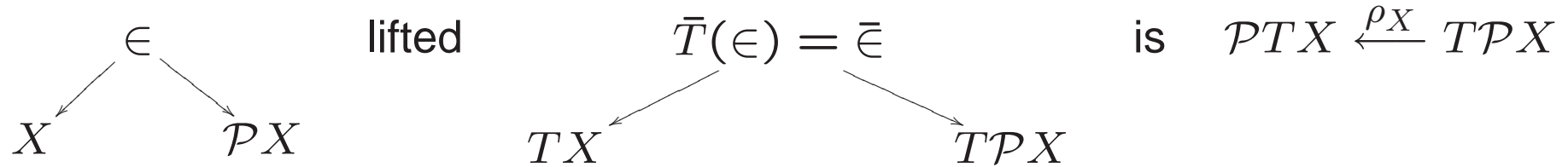
Yde's talk: T_ω is the finitary version of T , technically $T_\omega X = \bigcup \{TY \mid Y \subseteq_\omega X\}$.

Example: $\mathcal{P}_\omega X$ is the set of all *finite* subsets of X

$$x \Vdash \nabla \alpha \Leftrightarrow (\xi(x), \alpha) \in \bar{T}(\Vdash)$$

Thm(Moss): \mathcal{L} is invariant under bisimulation. The (original) infinitary version characterises bisimilarity.

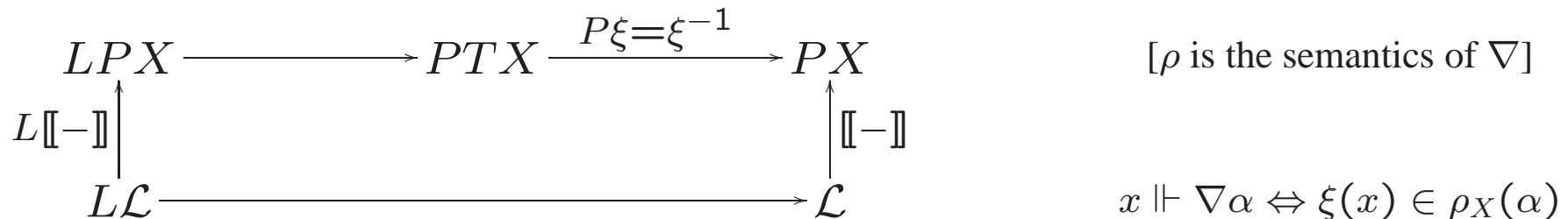
algebraic reformulation of the semantics



Define $L = FT_\omega U$ (where $U : \text{BA} \rightarrow \text{Set}$ and F its left-adjoint).

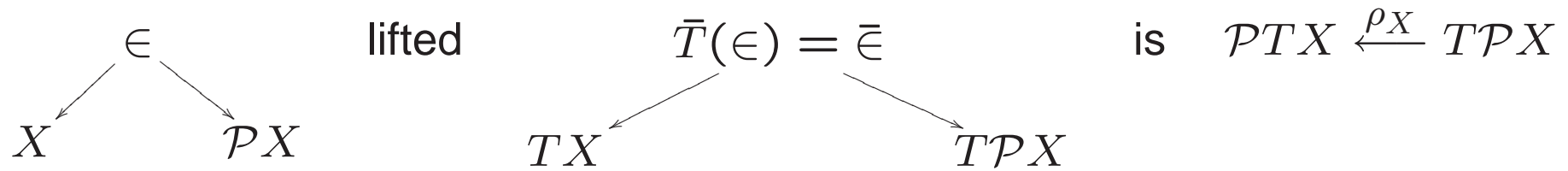
$\rho : T\mathcal{P}X \rightarrow \mathcal{P}TX$ induces a BA-morphism $L\mathcal{P}X \rightarrow \mathcal{P}TX$.

The semantics of \mathcal{L} wrt $\xi : X \rightarrow TX$ is given by the ‘complex algebra’ of X :



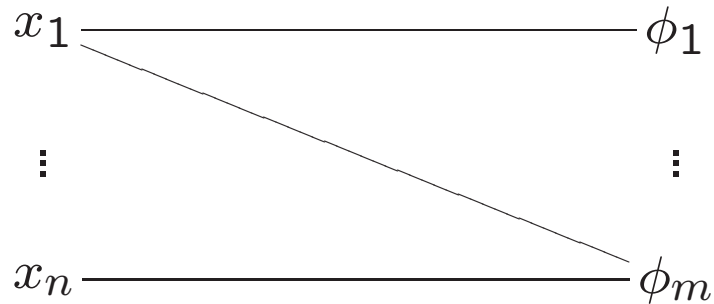
Notation: $\mathcal{P} : \text{Set} \rightarrow \text{Set}$, $P : \text{Set}^{\text{op}} \rightarrow \text{Set}$, $\mathbb{P} : \text{Set}^{\text{op}} \rightarrow \text{BA}$

examples

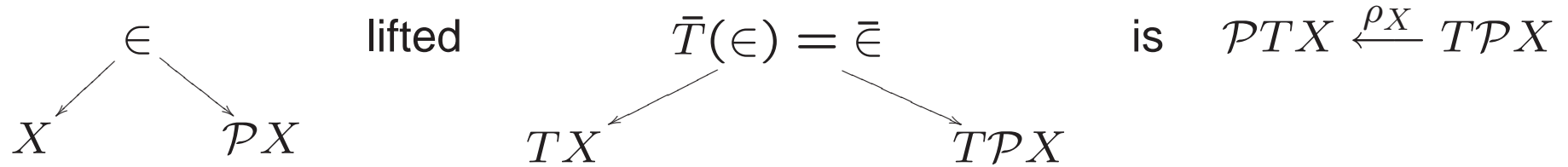


Examples for $\psi \bar{\epsilon} \Phi$ or $\psi \in \rho_X(\Phi)$:

$T = \mathcal{P}$: $\forall x \in \psi. \exists \phi \in \Phi. x \in \phi$ and vice versa

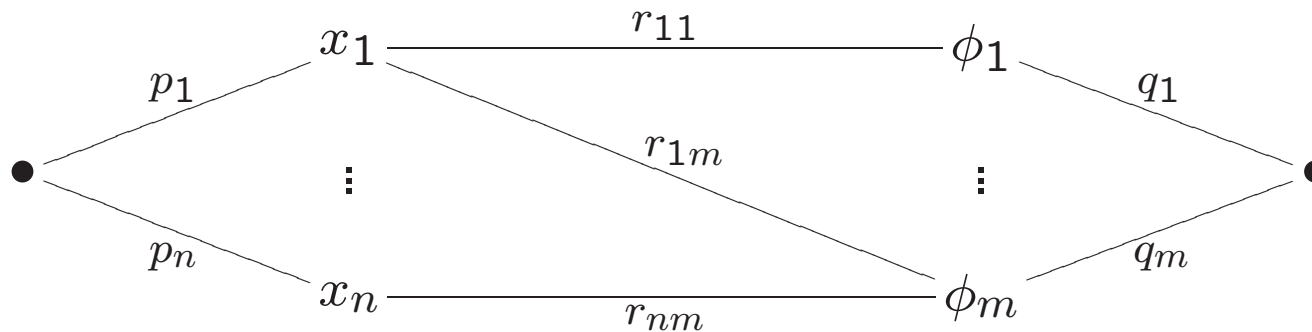


examples



Examples for $\psi \bar{\epsilon} \Phi$ or $\psi \in \rho_X(\Phi)$:

$$TX = \{d : X \rightarrow [0, 1] \mid d(x) = 0 \text{ almost everywhere} \}$$



the proof system (finite case)

Notation:

\mathcal{L}	a, b, c, \dots	$T_\omega \mathcal{L}$	$\alpha, \beta, \gamma \dots$
$\mathcal{P}_\omega \mathcal{L}$	ϕ, ψ, \dots	$T_\omega \mathcal{P}_\omega \mathcal{L}$	Φ, Ψ, \dots
$\mathcal{P}_\omega T_\omega \mathcal{L}$	$A, B, C \dots$		

If T preserves finite sets (maps finite sets to finite sets):

- | | |
|----------------|--|
| ($\nabla 1$) | From $\alpha \bar{\preceq} \beta$ infer $\vdash \nabla \alpha \preceq \nabla \beta$ |
| ($\nabla 2$) | $\bigwedge \{ \nabla \alpha \mid \alpha \in A \} \preceq \bigvee \{ \nabla (T \wedge)(\Phi) \mid \Phi \in SRD(A) \}$ |
| ($\nabla 3$) | $\nabla (T \vee)(\Phi) \preceq \bigvee \{ \nabla \alpha \mid \alpha \bar{\in} \Phi \}$ |

the proof system (infinitary rules)

For an arbitrary (weak pullback preserving) functor $T : \text{Set} \rightarrow \text{Set}$

$$(\nabla 1) \frac{\{b_1 \preceq b_2 \mid (b_1, b_2) \in Z\}}{\nabla \alpha \preceq \nabla \beta} (\alpha, \beta) \in \bar{Z}$$

$$(\nabla 2) \frac{\{\nabla(T \wedge)(\Phi) \preceq a \mid \Phi \in SRD(A)\}}{\wedge \{\nabla \alpha \mid \alpha \in A\} \preceq a}$$

$$(\nabla 3) \frac{\{\nabla \alpha \preceq a \mid \alpha \in \bar{\Phi}\}}{\nabla(T \vee)(\Phi) \preceq a}$$

reminder: completeness of \mathbf{K}

(in the style of Abramsky's Domain Theory in Logical Form)

$$\text{BA} \begin{array}{c} \xrightarrow{\mathbb{P}} \\ \xleftarrow{\mathbb{S}} \end{array} \text{Set} \quad (1)$$

Define $\mathbb{K} : \text{BA} \rightarrow \text{BA}$ as follows:

$\mathbb{K}(\mathbb{A})$ is generated by $\Box a, a \in \mathbb{A}$, modulo $\Box(a \wedge b) = \Box a \wedge \Box b, \Box \top = \top$.

Note: Every 'variable' a, b is under the scope of exactly one modality

Thm: $\mathbb{K}\mathbb{P}X \rightarrow \mathbb{P}\mathcal{P}X, \Box a \mapsto \{b \subseteq a\}$, is an isomorphism for finite sets X .

Cor: a) One-step completeness: $\mathbb{K}\mathbb{P}X \rightarrow \mathbb{P}\mathcal{P}X$ is injective for all X .

b) Completeness of \mathbf{K} .

remark on 'one-step completeness'

(Now writing T for \mathcal{P})

Show

$\mathbb{K}PX \rightarrow \mathbb{P}TX$ injective

or

$TX \rightarrow S\mathbb{K}PX$ surjective

\mathbb{M} and the one-step proof system

What is the analog of \mathbb{K} in our case?

Define $\mathbb{M} : BA \rightarrow BA$

$\mathbb{M}\mathbb{A}$ is given by

generators: $\nabla\alpha, \alpha \in T_\omega U\mathbb{A}$

modulo: $(\nabla 1)$ - $(\nabla 3)$

In the paper we make precise what we mean by ‘modulo’ here: we call it the *one-step proof system*

final coalgebra sequence and initial algebra sequence

$$1 \longleftarrow \mathcal{P}1 \longleftarrow \dots \longleftarrow \mathcal{P}^n 1 \longleftarrow \dots \longleftarrow \mathcal{P}^\omega 1 \quad \text{canonical model}$$

$$2 \longrightarrow \mathbb{K}2 \longrightarrow \dots \longrightarrow \mathbb{K}^n 2 \longrightarrow \dots \longrightarrow \mathbb{K}^\omega 2 \quad \text{Lindenbaum algebra}$$

[two references: Abramsky'89, Ghilardi'95]

final coalgebra sequence and initial algebra sequence

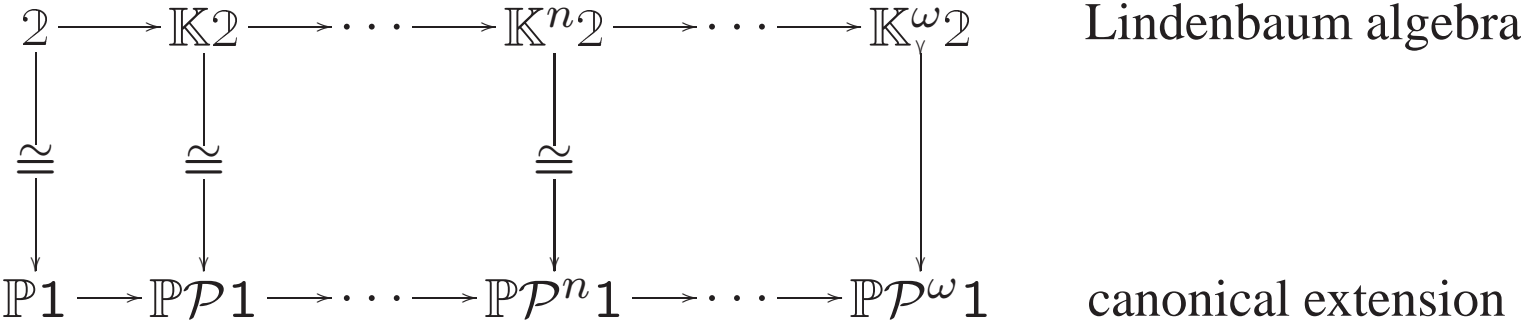
$$1 \longleftarrow \mathcal{P}1 \longleftarrow \dots \longleftarrow \mathcal{P}^n 1 \longleftarrow \dots \longleftarrow \mathcal{P}^\omega 1 \quad \text{canonical model}$$

$$2 \longrightarrow \mathbb{K}2 \longrightarrow \dots \longrightarrow \mathbb{K}^n 2 \longrightarrow \dots \longrightarrow \mathbb{K}^\omega 2 \quad \text{Lindenbaum algebra}$$

$$\mathbb{P}1 \longrightarrow \mathbb{P}\mathcal{P}1 \longrightarrow \dots \longrightarrow \mathbb{P}\mathcal{P}^n 1 \longrightarrow \dots \longrightarrow \mathbb{P}\mathcal{P}^\omega 1 \quad \text{canonical extension}$$

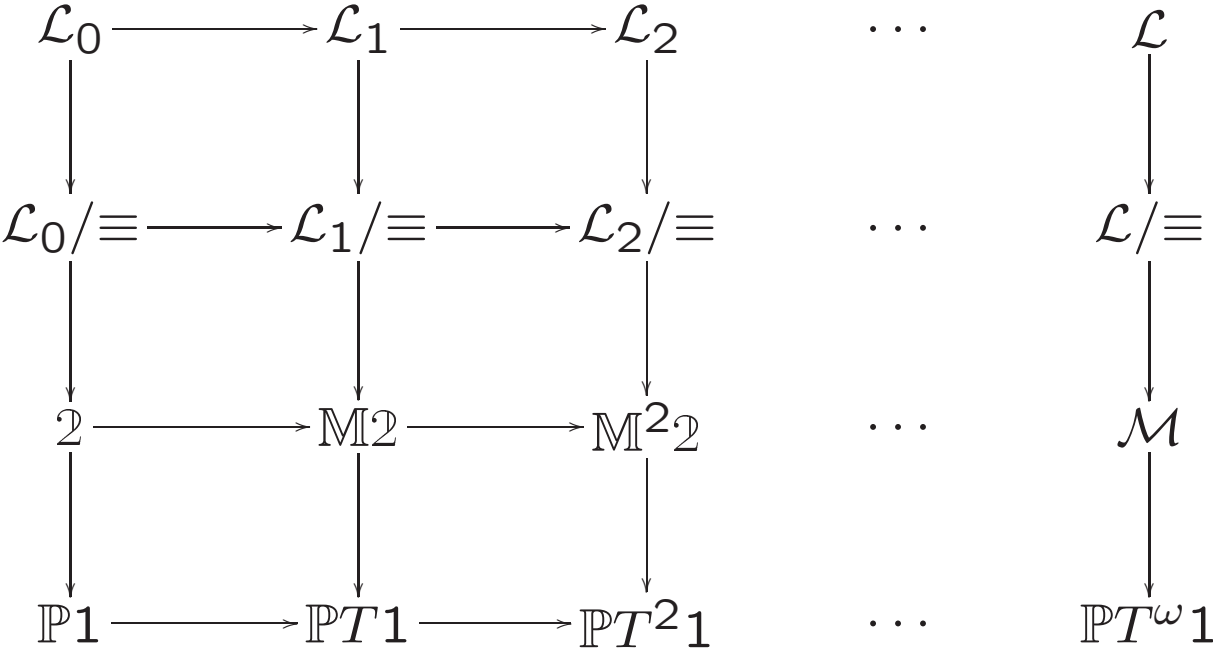
final coalgebra sequence and initial algebra sequence

$$1 \longleftarrow \mathcal{P}1 \longleftarrow \dots \longleftarrow \mathcal{P}^n 1 \longleftarrow \dots \longleftarrow \mathcal{P}^\omega 1 \quad \text{canonical model}$$



from one-step completeness to completeness

\mathcal{L} : Moss's language, \mathcal{L}_i : formulas of depth i



from one-step completeness to completeness needs ...

$\mathcal{L}_n / \equiv \longrightarrow \mathbb{M}^n \mathcal{L}$ is iso [Derivations of $\vdash a \equiv b$ of terms a, b of depth n can be performed without using terms of depth $> n$. Follows from the fact that the logic is described by a one-step proof system.]

\mathbb{M} is a functor [Given a BA-morphism $f : \mathbb{A} \rightarrow \mathbb{B}$, a derivation of $a \equiv a'$ in the one-step proof system over \mathbb{A} can be mapped to a derivation of $f(a) \equiv f(a')$ in the one-step proof system over \mathbb{B} .]

\mathbb{M} is finitary and preserves embeddings [Given an injective BA-morphism $f : \mathbb{A} \rightarrow \mathbb{B}$, a derivation of $f(a) \equiv f(a')$ in the one-step proof system over \mathbb{B} can be mapped to a derivation of $a \equiv a'$ in the one-step proof system over \mathbb{A} (proof uses that for a finite BA \mathbb{A} an embedding $\mathbb{A} \rightarrow \mathbb{B}$ has a half-inverse (which follows eg from the fact that complete Boolean algebras are injective))]

one-step completeness

Show $\delta : \mathbb{M}PX \rightarrow \mathbb{P}TX$ is injective. Idea: Find a half-inverse.

How can we go from $\mathbb{P}TX$ to $\mathbb{M}PX$? Recall: $\mathbb{M}PX$ is generated by elements in TPX (and $\mathbb{P}TX$ is generated by elements in TX).

So we need $TX \rightarrow TPX$, which is provided by applying T to $\{\} : X \rightarrow PX$.

So let $\mathcal{G} = \{\nabla(T\{\})(\alpha) \mid \alpha \in T_\omega X\}$. Note that $\delta(\nabla(T\{\})(\alpha)) = \{\alpha\}$.

We have to show $\forall a \in \mathbb{M}PX. a = \bigvee \{\nabla\beta \in \mathcal{G} \mid \nabla\beta \leq a\}$.

Case 1: $a = \nabla\beta, \beta \in T_\omega PX$. Uses ($\nabla 3$): $\nabla(T\bigvee)(\Phi) \preceq \bigvee \{\nabla\alpha \mid \alpha \in \Phi\}$

Case 2: $a = \neg\nabla\beta$. Uses ($\nabla 4$): From $\vdash \top \preceq \bigvee \phi$ infer $\vdash \top \preceq \bigvee \{\nabla\alpha \mid \alpha \in T\phi\}$

Case 3: $a = \bigwedge \beta_i$. Uses ($\nabla 2$): $\bigwedge \{\nabla\alpha \mid \alpha \in A\} \preceq \bigvee \{\nabla(T\bigwedge)(\Phi) \mid \Phi \in SRD(A)\}$

(my) conclusion

Given a category \mathcal{X} and a functor $T : \mathcal{X} \rightarrow \mathcal{X}$, what can we say about logics for T -coalgebras? [in the talk: $\mathcal{X} = \text{Set}$.]

What is the propositional ‘base logic’? Choose a category \mathcal{A} of algebras with appropriate $P : \mathcal{X} \rightarrow \mathcal{A}$ [in the talk: $\mathcal{A} = \text{BA}$, P powerset.]

Extend the base logic by modal operators and axioms: choose a functor $L : \text{BA} \rightarrow \text{BA}$ and semantics $\delta : LP \rightarrow PT$ [δ induces map $\text{Coalg}(T) \rightarrow \text{Alg}(L)$.]

One of the strength of this approach is that it is parametric in the base categories. For example, we want to look at (future work):

\mathcal{A} could be distributive lattices, complete atomic Boolean algebras

\mathcal{X} could be posets