Axiomatising many-dimensional modal logics

Agi Kurucz

Department of Computer Science

King's College London

Axiomatising products of modal logics

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Department of Computer Science

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Axiomatising multi-modal logics

- **n-modal formulas:** $\boxed{\varphi \; = \; p \mid \neg\varphi \mid \varphi_1 \vee \varphi_2 \mid \Diamond_0\varphi \mid \cdots \mid \Diamond_{n-1}\varphi}$

- **(normal) n-modal logic:** a set of n-modal formulas
  - containing the axioms **K**$_i$ $\boxed{\Box_i(p \to q) \to (\Box_i p \to \Box_i q)}$
  - closed under the rules of **Substitution**, **Modus Ponens**, and **Necessitation**$_i$ $\boxed{\varphi \; / \; \Box_i\varphi}$ for all $i < n$.

Example: $\boxed{\textbf{Logic\_of } \mathcal{C} \; = \; \{\varphi \mid \forall \mathfrak{F} \in \mathcal{C},\; \mathfrak{F} \models \varphi\}}$ for any class $\mathcal{C}$ of n-frames.

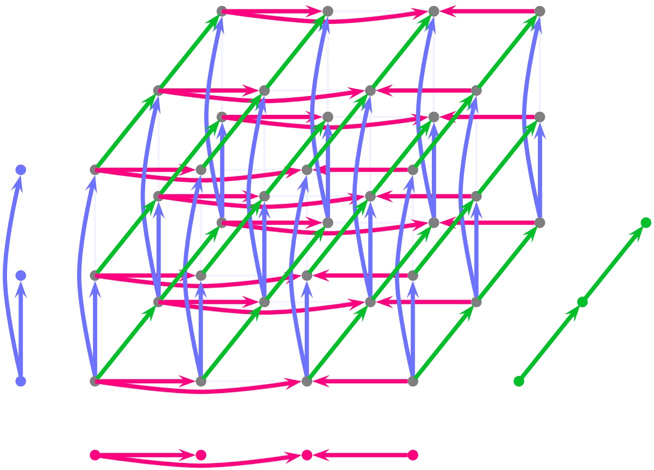
- A recursive set Σ of n-modal formulas **axiomatises** an n-modal logic L, if L is the smallest logic containing Σ.

Product of Kripke frames

Given frames $\mathfrak{F}_0 = (U_0, R_0), \quad \ldots \quad \mathfrak{F}_{n-1} = (U_{n-1}, R_{n-1})$, their **product** is

$$\mathfrak{F}_0 \times \cdots \times \mathfrak{F}_{n-1} = (U_0 \times \cdots \times U_{n-1}, \overline{R}_0, \ldots \overline{R}_{n-1}) \quad \text{where}$$

$$(u_0, \ldots u_{n-1}) \overline{R}_i (v_0, \ldots v_{n-1}) \quad \text{iff} \quad u_i R_i v_i \text{ and } u_j = v_j \quad \text{for } j \neq i < n$$



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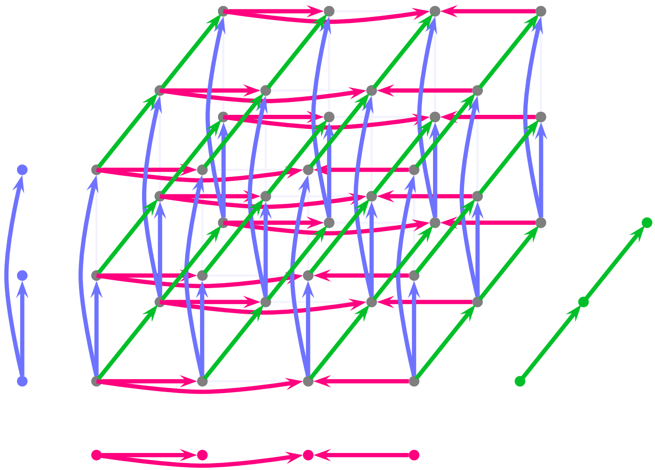

- Any **point-generated subframe** of a product frame is a product frame.
- Any **ultraproduct** of product frames is isomorphic to a product frame.

Product of modal logics

Segerberg 1973, Shehtman 1978, Gabbay-Shehtman 1998

Given Kripke complete unimodal logics $L_0, \ \ldots \ L_{n-1}$, their **product** is

$$\boxed{L_0 \times \cdots \times L_{n-1} \ = \ \textbf{Logic\_of}\ \{\mathfrak{F}_0 \times \cdots \times \mathfrak{F}_{n-1} \mid \mathfrak{F}_i \text{ is an } L_i\text{-frame, for } i < n\}}$$

- K^n = **Logic_of** {all n-ary product frames}
- $\mathrm{K4}^n$ = **Logic_of** {all n-ary products of transitive frames}
- $\mathrm{S5}^n$ = **Logic_of** {all n-ary products of equivalence frames}
- Alt^n = **Logic_of** {all n-ary products of functional frames}
- $\mathrm{K4} \times \mathrm{GL.3}$ = **Logic_of** {$\mathfrak{F}_0 \times \mathfrak{F}_1$ | both $\mathfrak{F}_i$ are transitive, and $\mathfrak{F}_1$ is weakly connected and has no infinite ascending chains}

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- L is **recursively enumerable**, whenever the class of all L_i-frames has a recursive first-order axiomatisation, for each $i < n$.

  Not always: $\mathbf{K4} \times \mathbf{GL.3}$ $\quad$ $\mathbf{K4} \times$ **Logic_of** $\{(\omega, <)\}$ — Π^1_1**-hard**

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- L is **canonical**, whenever the class of all L_i-frames is elementary, for each $i < n$.

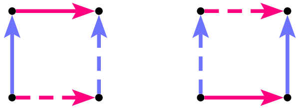
Axiomatising product logics: first steps

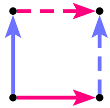
Given Kripke complete unimodal logics $L_0, \ldots L_{n-1}$, their **commutator**

$$\boxed{[L_0, \ldots L_{n-1}]}$$

is the smallest n-modal logic containing L_i (for $\Diamond_i$) and the (Sahlqvist) formulas

$$\boxed{\Box_i \Box_j p \leftrightarrow \Box_j \Box_i p} \quad \text{and} \quad \boxed{\Diamond_i \Box_j p \rightarrow \Box_j \Diamond_i p} \quad \text{for all } i \neq j < n.$$





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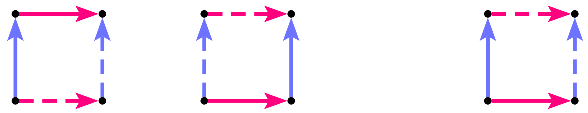

- $[L_0, \dots L_{n-1}] \subseteq L_0 \times \cdots \times L_{n-1}$

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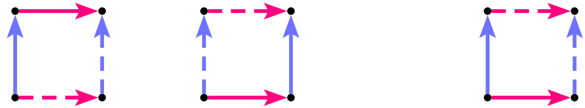

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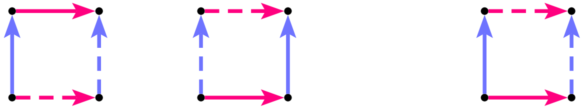

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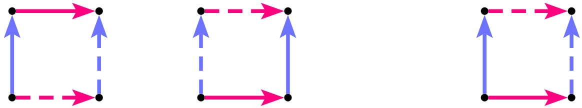

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$$\boxed{[L_0, \ldots L_{n-1}] \stackrel{?}{=} L_0 \times \cdots \times L_{n-1}}$$

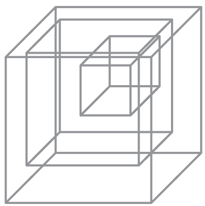
- Not even when both logics are canonical: $[\mathbf{K4}, \mathbf{K4.3}] \neq \mathbf{K4} \times \mathbf{K4.3}$

  Open: finite axiomatisability of $\mathbf{K} \times \mathbf{K4.3}$ $\quad \mathbf{K4} \times \mathbf{K4.3}$ $\quad \mathbf{S5} \times \mathbf{K4.3}$ $\quad \ldots$

Trying to axiomatise $\mathbf{K}^n$

Given a **countable, rooted** n-frame $\boxed{\mathfrak{F}}$ validating the suggested axioms, we try to obtain it as a **p-morphic image of a product frame**, step by step:

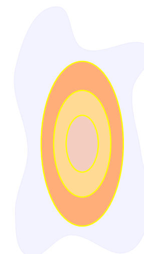
$(U_0 \times \cdots \times U_{n-1}, \overline{R}_0, \ldots \overline{R}_{n-1})$



$\bar{u}\overline{R}_i\bar{v} \implies f(\bar{u})S_i f(\bar{v})$

$\xrightarrow{\quad f \quad}$

$\mathfrak{F} = (W, S_0, \ldots S_{n-1})$



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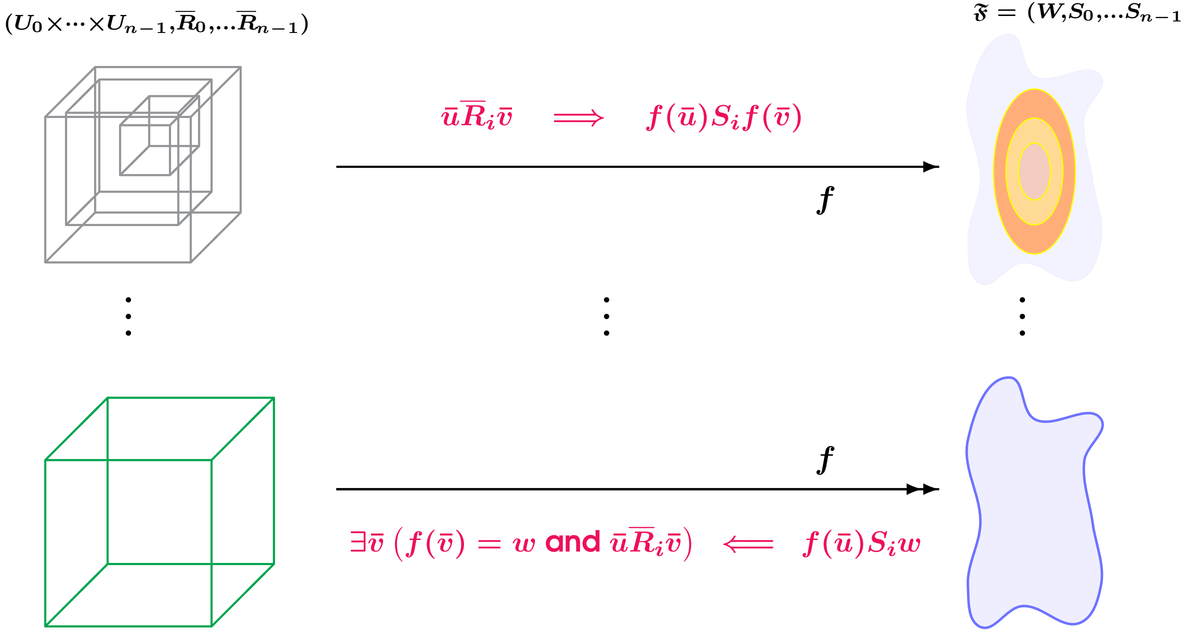
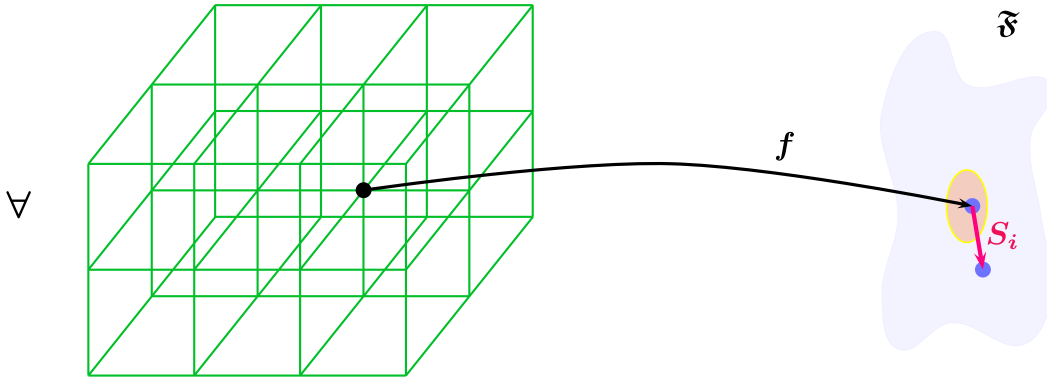


The "p-morphism" game



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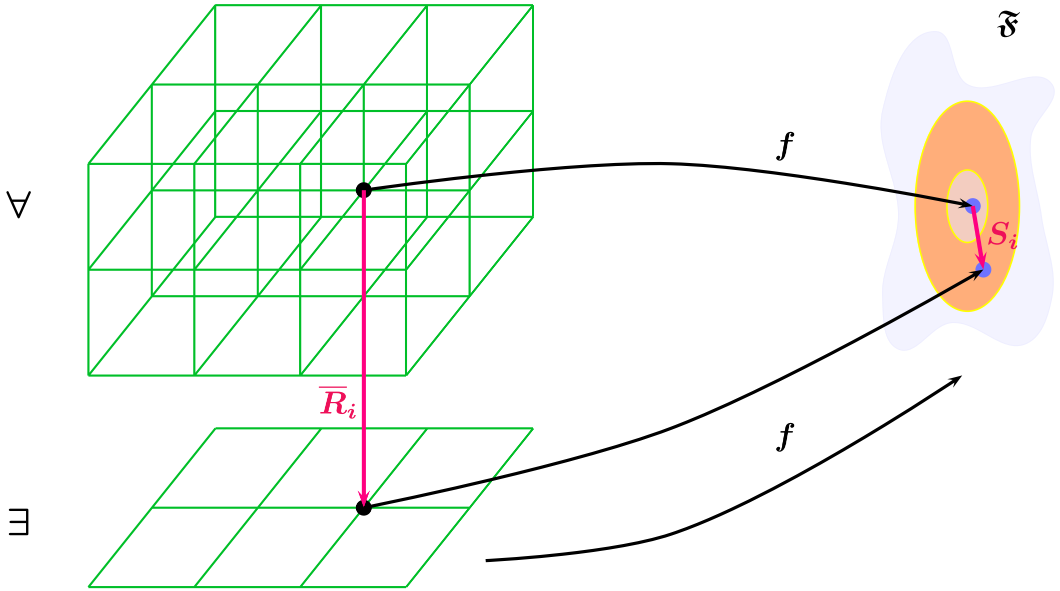

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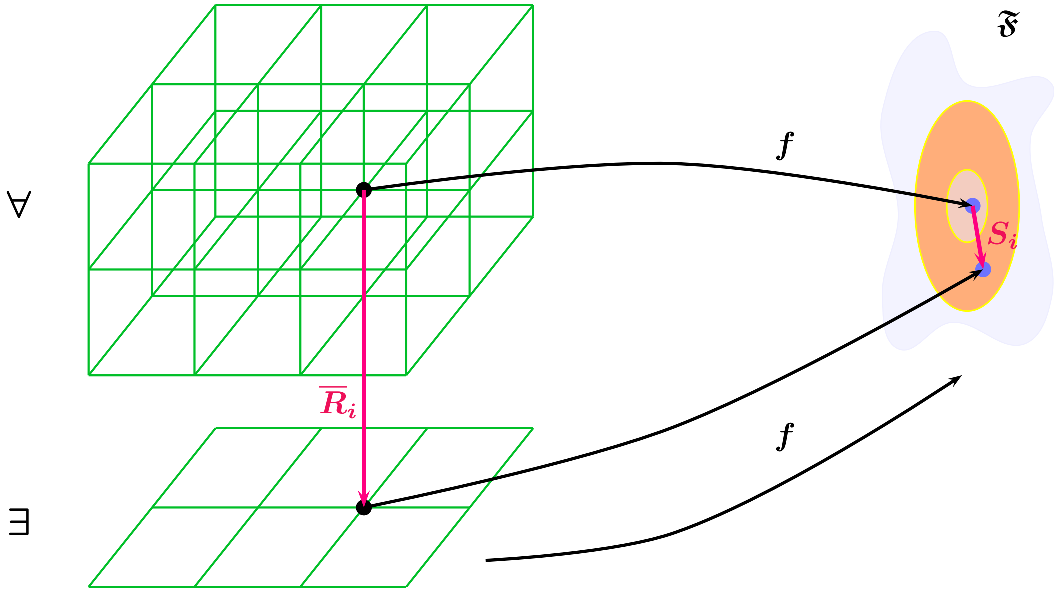

$\exists$ has a winning strategy over $\mathfrak{F}$ (when $\forall$ begins with its root) $\iff$ $\mathfrak{F}$ is a p-morphic image of a product frame

And other product logics?

- **Horn formula:** $\boxed{\forall xy\bar{z}\,\big(\Phi(x,y,\bar{z}) \to R(x,y)\big)}$ where Φ is positive
- **Horn axiomatisable modal logic:** axiomatisable by modal formulas having Horn first-order correspondents $\mathbf{K},\ \mathbf{K4},\ \mathbf{S4},\ \mathbf{S5},\ \ldots$
- For $i < n$, let L_i be Horn axiomatisable, and $\mathfrak{F}$ an n-frame such that:
  - $\mathfrak{F} \models$ 'L_i for $\Diamond_i$', and
  - $\exists$ onto p-morphism $f : \mathfrak{G}_0 \times \cdots \times \mathfrak{G}_{n-1} \to \mathfrak{F}$,

  then $f : \mathfrak{G}_0^* \times \cdots \times \mathfrak{G}_{n-1}^* \to \mathfrak{F}$ is also a p-morphism, where each $\mathfrak{G}_i^*$ is the 'L_i-closure' of $\mathfrak{G}_i$.

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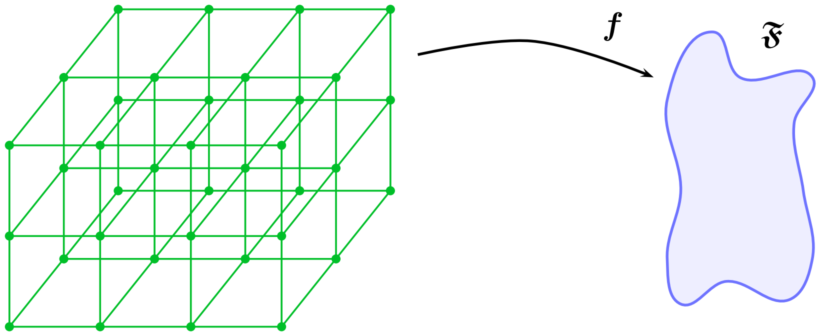

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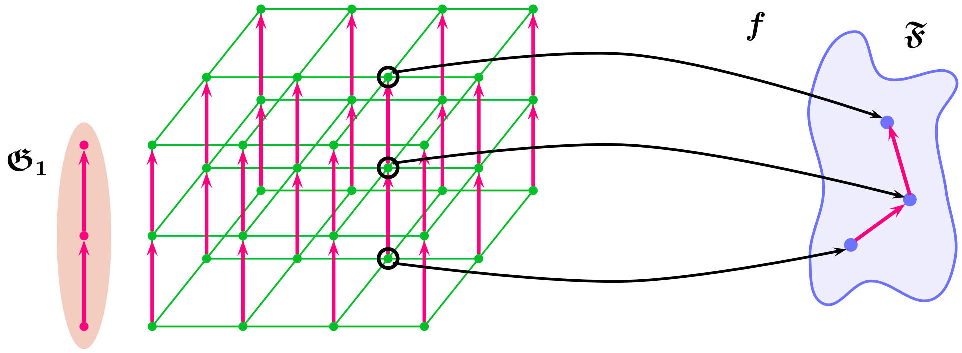

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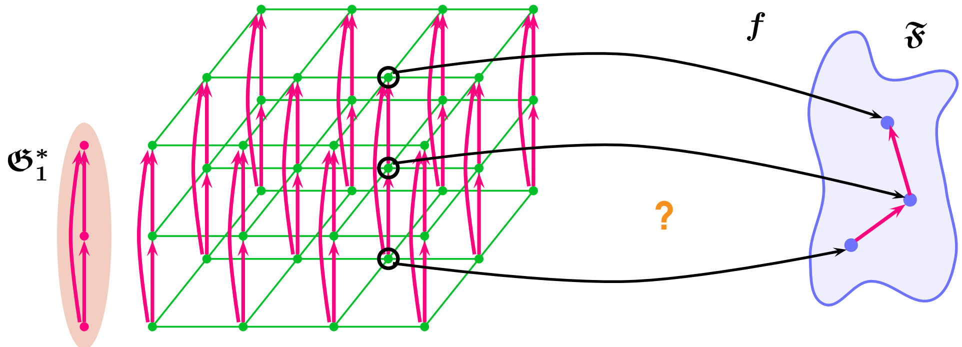

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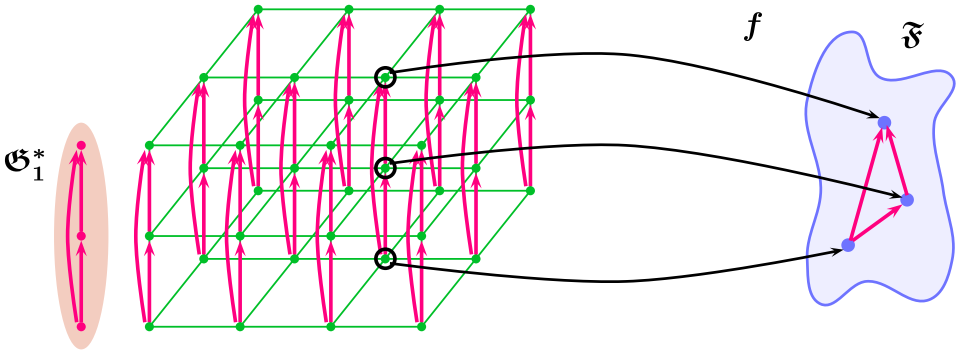

When the “p-morphism” game helps in axiomatising products

Gabbay-Shehtman 1998

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- $\boxed{[\mathbf{Alt}, \dots \mathbf{Alt}] = \mathbf{Alt}^n}$ for any $n < \omega$

  commutativity and confluence are again enough for defining a winning strategy for $\exists$ in the p-morphism game

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Moreover:

- Modal algebras for $\mathbf{S5}^n$:
  RDf_n — **representable diagonal-free cylindric algebras of dimension n**
  - *Johnson 1969*
    $\boldsymbol{RDf_n}$ has a non-finitely axiomatisable equational theory
  - equations valid in $\boldsymbol{RDf_n}$ $\leftrightarrow$ $\boldsymbol{n}$-modal formulas valid in $\mathbf{S5}^{\boldsymbol{n}}$

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  - it is **undecidable** whether a finite $\boldsymbol{n}$-frame is an $\boldsymbol{L}$-frame

$\mathrm{S5}^n$ plus diagonals

Diagonal constants d_{ij}, for $i, j < n$: $\boxed{d_{ij}^{\mathfrak{F}_0 \times \dots \times \mathfrak{F}_{n-1}} = \{(u_0, \dots u_{n-1}) \mid u_i = u_j\}}$

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Modal algebras for '$\mathbf{S5}^n$ plus diagonals': **representable cylindric algebras**

- *Monk 1969*

  '$\mathbf{S5}^n$ plus diagonals' is **non-finitely axiomatisable**
- *Monk 1969, Hirsch-Hodkinson 1997*

  Explicit infinite axiomatisation for '$\mathbf{S5}^n$ plus diagonals'
- *Andréka 1997*

  Any axiomatisation for '$\mathbf{S5}^n$ plus diagonals' must contain

  - **infinitely many propositional variables**
  - **infinitely many occurrences of each diagonal constant**
- *Hodkinson 1997, Venema 1997*

  '$\mathbf{S5}^n$ plus diagonals' has no axiomatisation with **Sahlqvist** formulas
- *Hodkinson-Venema 2005*

  Every first-order axiomatisation for **representable relation algebras** must contain **infinitely many non-canonical formulas**

  Most probably a similar result holds for representable cylindric algebras

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- $\boldsymbol{RDf_n} = \mathbb{HSP}\{\boldsymbol{Cm}(\boldsymbol{\mathfrak{F}_0} \times \cdots \times \boldsymbol{\mathfrak{F}_{n-1}}) \mid \boldsymbol{\mathfrak{F}_i}$ are $\mathbf{S5}$-frames$\}$

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  - $\boldsymbol{RDf_n}$ is a **discriminator** variety ($= \mathbf{S5}^n$ has **universal modality**)

    $\rightsquigarrow$ universal formulas are equivalent to equations

    $\rightsquigarrow$ **explicit infinite axiomatisation for $\mathrm{S5}^n$**

Dissecting the non-finite axiomatisability of $\mathrm{S5}^n$

How much of this is due to different factors:

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$\rightsquigarrow$ the many-dimensional structure is responsible for the complexity

$\rightsquigarrow$ negative axiomatisation results for $\mathbf{S5}^n \implies$ same for $\mathbf{K}^n$, $\mathbf{K4}^n$, $\mathbf{S4}^n$, $\dots$

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  - **But there is no universal modality in an arbitrary product frame**

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- Is there an axiomatisation for $\mathbf{S5}^n$ **using finitely many propositional variables**?

$\mathbf{K}^n$ is not axiomatisable using finitely many variables

If

- If L contains $\mathbf{K}^n$, and
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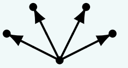

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Examples: $\mathbf{K}^n$ $\mathbf{K4}^n$ $\mathbf{S4}^n$ $\mathbf{GL}^n$ $\ldots$ (but not $\mathbf{S5}^n$)

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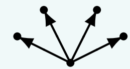

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Proof:

For every $k < \omega$, we give a (finite) n-frame $\boxed{\mathfrak{F}_k}$ such that:

- $\mathfrak{F}_k$ is not a $\mathbf{K}^n$-frame, but
- every m-generated model based on $\mathfrak{F}_k$ is a model for L, if $k > 2^{m-1}$

How to show that $\mathfrak{F}_k$ is not a K^n-frame

For every $k < \omega$, we give an 3-modal (Sahlqvist) formula $\boxed{\varphi_k}$ such that:

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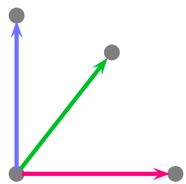
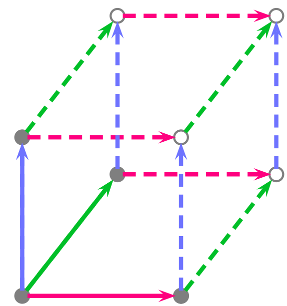


$\exists$



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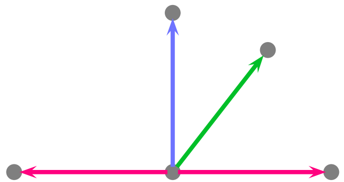

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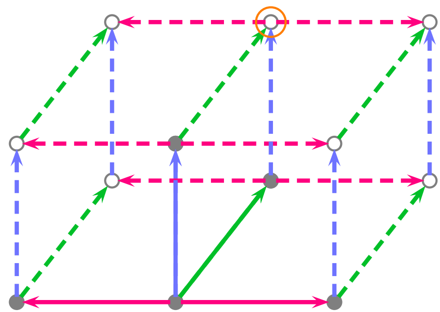

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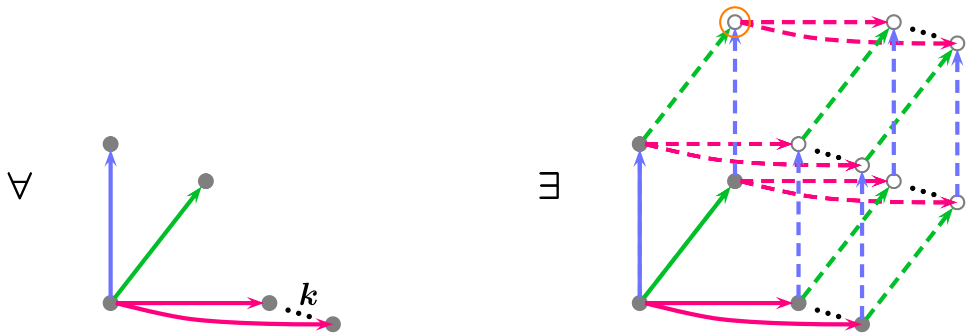

How to have control over 'small generated' models over $\mathfrak{F}_k$

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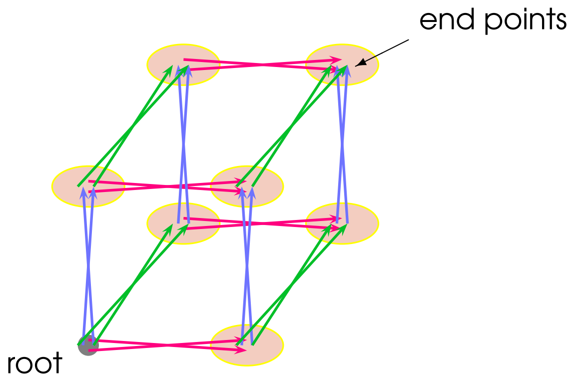

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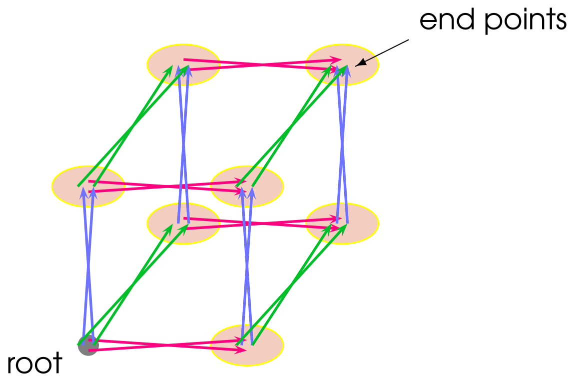

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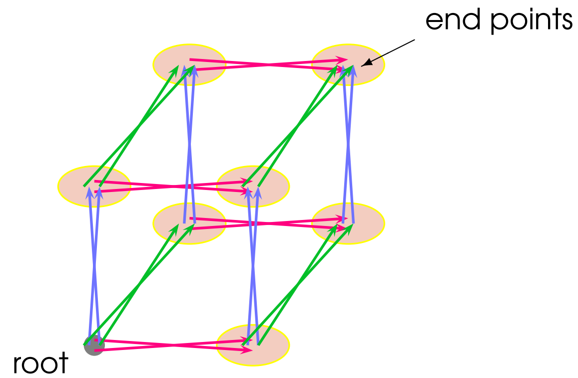

- every p-morphic image of $\mathfrak{F}_k$ that unifies different end-points is a p-morphic image of a **product of finite depth-one fans**
- few propositional variables can partition end-points to few blocks only

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