

Axiomatising many-dimensional modal logics

Agi Kurucz

*Department of Computer Science
King's College London*

Axiomatising products of modal logics

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Axiomatising multi-modal logics

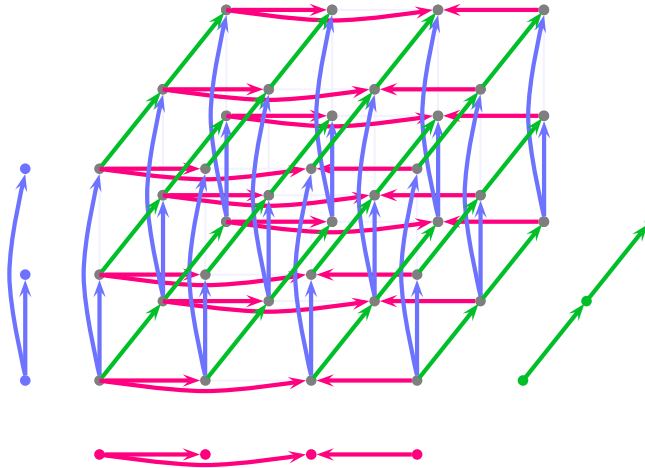
- **n -modal formulas:** $\varphi = p \mid \neg\varphi \mid \varphi_1 \vee \varphi_2 \mid \diamond_0\varphi \mid \dots \mid \diamond_{n-1}\varphi$
- **(normal) n -modal logic:** a set of n -modal formulas
 - containing the axioms \mathbf{K}_i $\Box_i(p \rightarrow q) \rightarrow (\Box_i p \rightarrow \Box_i q)$
 - closed under the rules of **Substitution**, **Modus Ponens**, and **Necessitation _{i}** $\varphi / \Box_i\varphi$ for all $i < n$.
- **Example:** **Logic of \mathcal{C}** = $\{\varphi \mid \forall \mathfrak{F} \in \mathcal{C}, \mathfrak{F} \models \varphi\}$ for any class \mathcal{C} of n -frames.
- A recursive set Σ of n -modal formulas **axiomatises** an n -modal logic \mathbf{L} , if \mathbf{L} is the smallest logic containing Σ .

Product of Kripke frames

Given frames $\mathfrak{F}_0 = (U_0, R_0), \dots, \mathfrak{F}_{n-1} = (U_{n-1}, R_{n-1})$, their **product** is

$$\mathfrak{F}_0 \times \dots \times \mathfrak{F}_{n-1} = (U_0 \times \dots \times U_{n-1}, \overline{R}_0, \dots, \overline{R}_{n-1}) \quad \text{where}$$

$$(u_0, \dots, u_{n-1}) \overline{R}_i (v_0, \dots, v_{n-1}) \quad \text{iff} \quad u_i R_i v_i \quad \text{and} \quad u_j = v_j \quad \text{for} \quad j \neq i < n$$

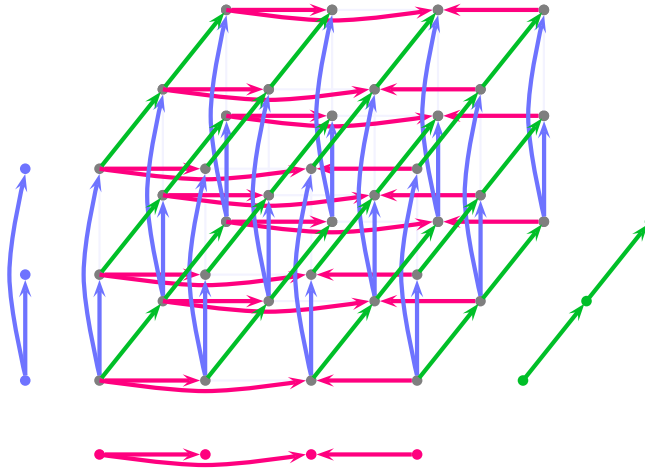


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- Any **point-generated subframe** of a product frame is a product frame.
- Any **ultraproduct** of product frames is isomorphic to a product frame.

Product of modal logics

Seegerberg 1973, Shehtman 1978, Gabbay-Shehtman 1998

Given Kripke complete unimodal logics L_0, \dots, L_{n-1} , their **product** is

$$L_0 \times \dots \times L_{n-1} = \mathbf{Logic_of} \{ \mathfrak{F}_0 \times \dots \times \mathfrak{F}_{n-1} \mid \mathfrak{F}_i \text{ is an } L_i\text{-frame, for } i < n \}$$

- $K^n = \mathbf{Logic_of} \{ \text{all } n\text{-ary product frames} \}$
- $K4^n = \mathbf{Logic_of} \{ \text{all } n\text{-ary products of transitive frames} \}$
- $S5^n = \mathbf{Logic_of} \{ \text{all } n\text{-ary products of equivalence frames} \}$
- $Alt^n = \mathbf{Logic_of} \{ \text{all } n\text{-ary products of functional frames} \}$
- $K4 \times GL.3 = \mathbf{Logic_of} \{ \mathfrak{F}_0 \times \mathfrak{F}_1 \mid \text{both } \mathfrak{F}_i \text{ are transitive, and } \mathfrak{F}_1 \text{ is weakly connected and has no infinite ascending chains} \}$

General properties of $L = L_0 \times \cdots \times L_{n-1}$

- L is determined by products of **rooted** L_i -frames.

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Not always: $K4 \times GL.3$ $K4 \times \text{Logic of } \{(\omega, <)\}$ — **Π_1^1 -hard**

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- L is **canonical**, whenever the class of all L_i -frames is elementary, for each $i < n$.

Axiomatising product logics: first steps

Given Kripke complete unimodal logics L_0, \dots, L_{n-1} , their **commutator**

$$[L_0, \dots, L_{n-1}]$$

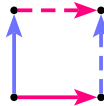
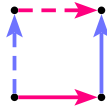
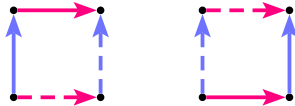
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$$\Box_i \Box_j p \leftrightarrow \Box_j \Box_i p$$

and

$$\Diamond_i \Box_j p \rightarrow \Box_j \Diamond_i p$$

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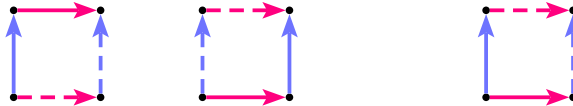
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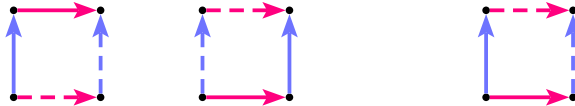
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- Not always even Kripke complete:** *Gabelaia-K-Wolter-Zakharyashev 2005*
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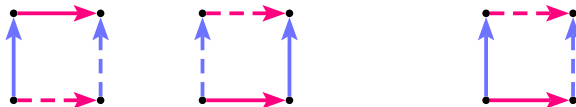
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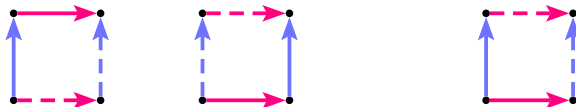
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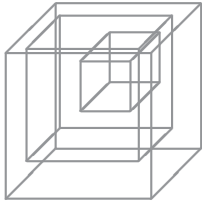
$$\boxed{[L_0, \dots, L_{n-1}] \stackrel{?}{=} L_0 \times \dots \times L_{n-1}}$$

- Not even when both logics are canonical: $[K4, K4.3] \neq K4 \times K4.3$
- **Open:** finite axiomatisability of $K \times K4.3$ $K4 \times K4.3$ $S5 \times K4.3$...

Trying to axiomatise K^n

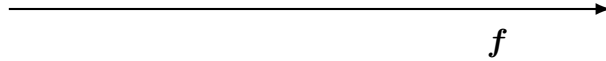
Given a **countable, rooted n -frame** \mathfrak{F} validating the suggested axioms, we try to obtain it as a **p-morphic image of a product frame**, step by step:

$$(U_0 \times \dots \times U_{n-1}, \bar{R}_0, \dots, \bar{R}_{n-1})$$



⋮

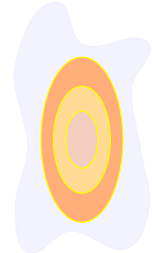
$$\bar{u} \bar{R}_i \bar{v} \implies f(\bar{u}) S_i f(\bar{v})$$



f

⋮

$$\mathfrak{F} = (W, S_0, \dots, S_{n-1})$$

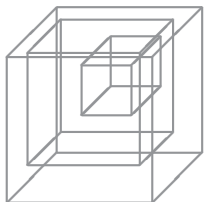


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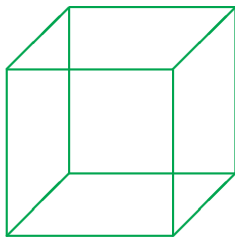
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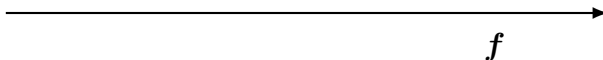
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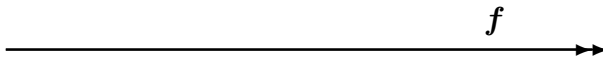


$$\bar{u} \bar{R}_i \bar{v} \implies f(\bar{u}) S_i f(\bar{v})$$



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$$\exists \bar{v} (f(\bar{v}) = w \text{ and } \bar{u} \bar{R}_i \bar{v}) \iff f(\bar{u}) S_i w$$



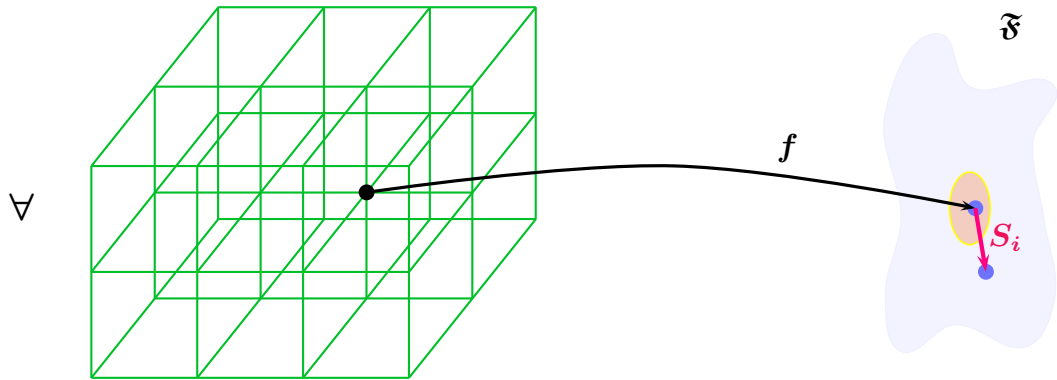
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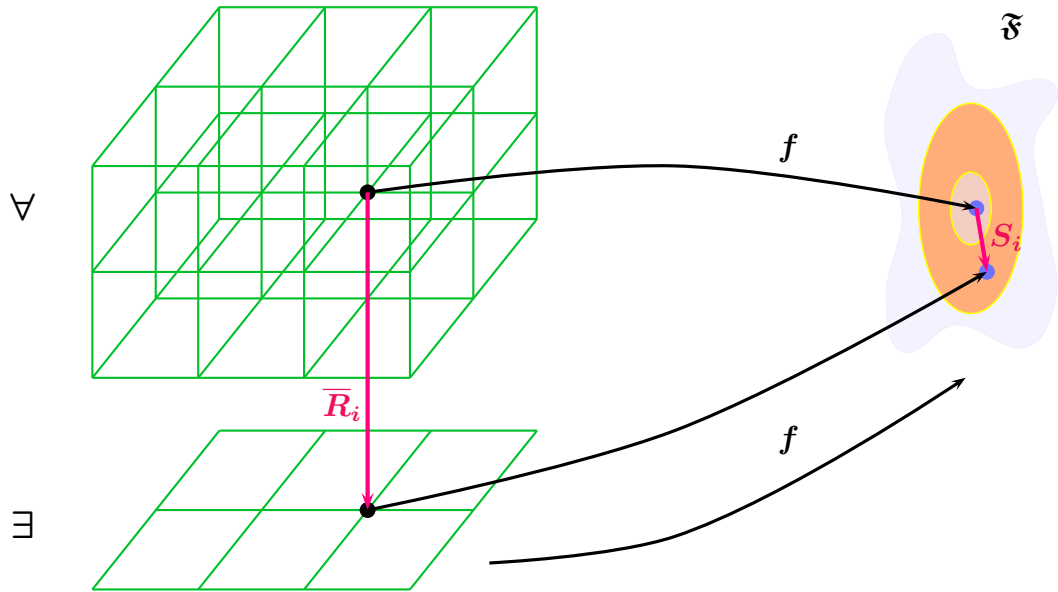
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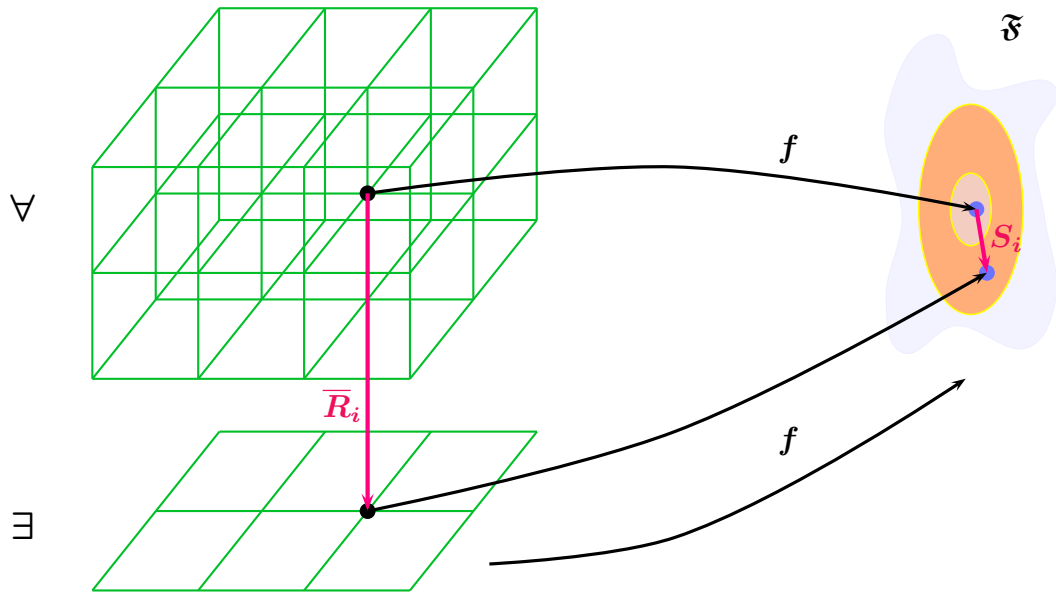
The “p-morphism” game



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\exists has a winning strategy over \mathfrak{F} (when \forall begins with its root)



\mathfrak{F} is a p-morphic image of a product frame

And other product logics?

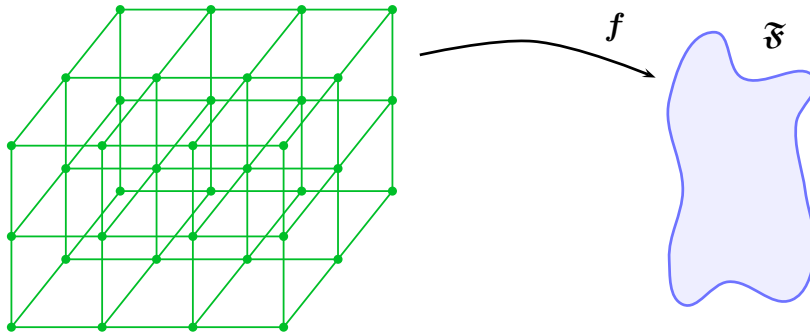
- **Horn formula:** $\forall xy\bar{z} (\Phi(x, y, \bar{z}) \rightarrow R(x, y))$ where Φ is positive
- **Horn axiomatisable modal logic:** axiomatisable by modal formulas having Horn first-order correspondents $\mathbf{K}, \mathbf{K4}, \mathbf{S4}, \mathbf{S5}, \dots$
- For $i < n$, let L_i be Horn axiomatisable, and \mathfrak{F} an n -frame such that:
 - $\mathfrak{F} \models L_i$ for \diamond_i , and
 - \exists onto p-morphism $f : \mathfrak{G}_0 \times \dots \times \mathfrak{G}_{n-1} \rightarrow \mathfrak{F}$,

then $f : \mathfrak{G}_0^* \times \dots \times \mathfrak{G}_{n-1}^* \rightarrow \mathfrak{F}$ is also a p-morphism, where each \mathfrak{G}_i^* is the ' L_i -closure' of \mathfrak{G}_i .

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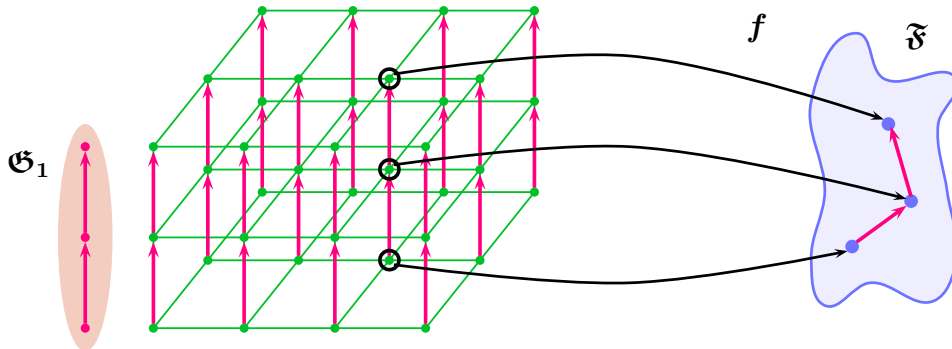
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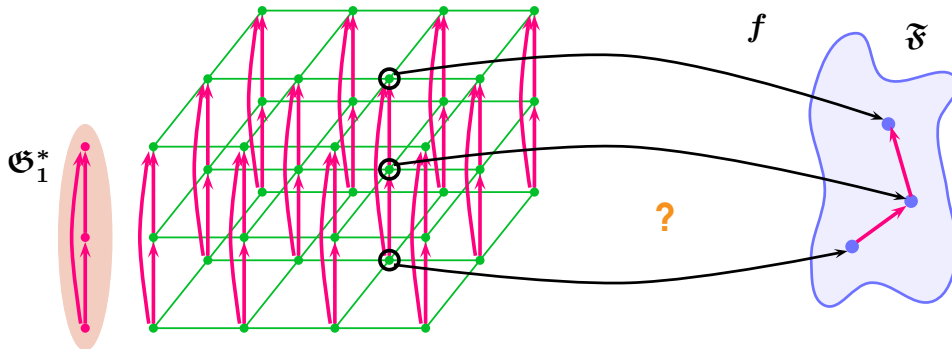
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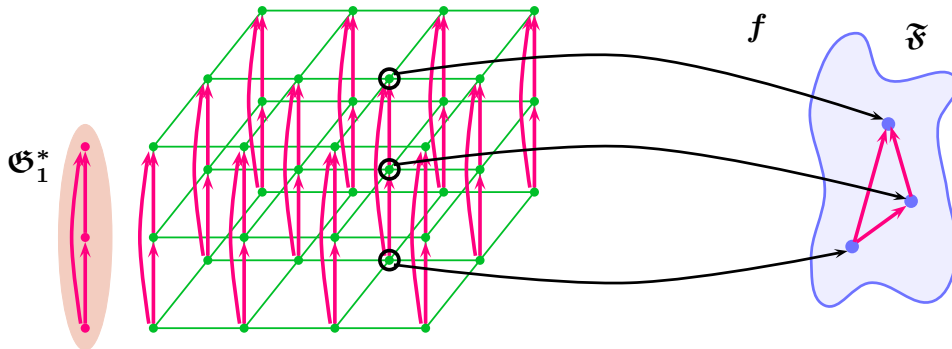
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When the “p-morphism” game helps in axiomatising products

Gabbay-Shehtman 1998

- $[K, K] = K \times K$

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- $[Alt, \dots, Alt] = Alt^n$ for any $n < \omega$

commutativity and confluence are again enough for defining a winning strategy for \exists in the p-morphism game

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Moreover:

- Modal algebras for $S5^n$:
 RDf_n — representable diagonal-free cylindric algebras of dimension n
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 RDf_n has a non-finitely axiomatisable equational theory
 - equations valid in $RDf_n \iff n$ -modal formulas valid in $S5^n$
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 - it is **undecidable** whether a finite n -frame is an L -frame

$S5^n$ plus diagonals

Diagonal constants d_{ij} , for $i, j < n$: $d_{ij}^{\tilde{s}_0 \times \dots \times \tilde{s}_{n-1}} = \{(u_0, \dots, u_{n-1}) \mid u_i = u_j\}$

Modal algebras for ' $S5^n$ plus diagonals': **representable cylindric algebras**

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Modal algebras for ' $S5^n$ plus diagonals': **representable cylindric algebras**

- *Monk 1969*

' $S5^n$ plus diagonals' is **non-finitely axiomatisable**

- *Monk 1969, Hirsch-Hodkinson 1997*

Explicit infinite axiomatisation for ' $S5^n$ plus diagonals'

- *Andréka 1997*

Any axiomatisation for ' $S5^n$ plus diagonals' must contain

- **infinitely many propositional variables**
- **infinitely many occurrences of each diagonal constant**

- *Hodkinson 1997, Venema 1997*

' $S5^n$ plus diagonals' has no axiomatisation with **Sahlqvist** formulas

- *Hodkinson-Venema 2005*

Every first-order axiomatisation for **representable relation algebras** must contain **infinitely many non-canonical formulas**

Most probably a similar result holds for representable cylindric algebras

Explicit (infinite) axiomatisation for $S5^n$

- $RDf_n = \text{HSP}\{Cm(\mathfrak{F}_0 \times \cdots \times \mathfrak{F}_{n-1}) \mid \mathfrak{F}_i \text{ are } S5\text{-frames}\}$

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- RDf_n is a **discriminator** variety (= $S5^n$ has **universal modality**)
 - \rightsquigarrow universal formulas are equivalent to equations
 - \rightsquigarrow **explicit infinite axiomatisation for $S5^n$**

Dissecting the non-finite axiomatisability of $S5^n$

How much of this is due to different factors:

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- \rightsquigarrow the many-dimensional structure is responsible for the complexity
- \rightsquigarrow negative axiomatisation results for $S5^n \implies$ same for $\mathbf{K}^n, \mathbf{K}4^n, S4^n, \dots$

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- **But there is no universal modality in an arbitrary product frame**

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- Is there an axiomatisation for $S5^n$ **using finitely many propositional variables**?

K^n is not axiomatisable using finitely many variables

If



- If L contains K^n , and
 - the product of n arbitrarily wide finite depth-one fans is a frame for L ,
- then L is **not axiomatisable using finitely many variables**.

Examples: K^n $K4^n$ $S4^n$ GL^n ... (but not $S5^n$)

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Proof:

For every $k < \omega$, we give a (finite) n -frame \mathfrak{F}_k such that:

- \mathfrak{F}_k is not a K^n -frame, but
- every m -generated model based on \mathfrak{F}_k is a model for L , if $k > 2^{m-1}$

How to show that \mathfrak{F}_k is not a K^n -frame

For every $k < \omega$, we give an $\mathfrak{3}$ -modal (Sahlqvist) formula φ_k such that:

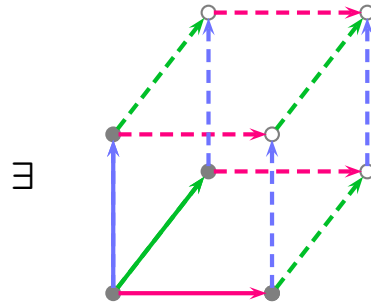
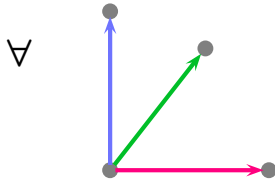
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How to show that \mathfrak{F}_k is not a \mathbb{K}^n -frame

For every $k < \omega$, we give an **3**-modal (Sahlqvist) formula φ_k such that:

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φ_1



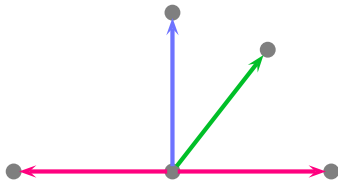
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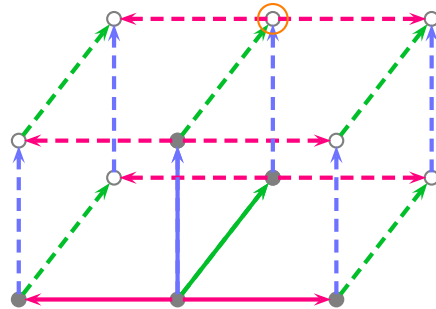
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φ_2

\forall



\exists

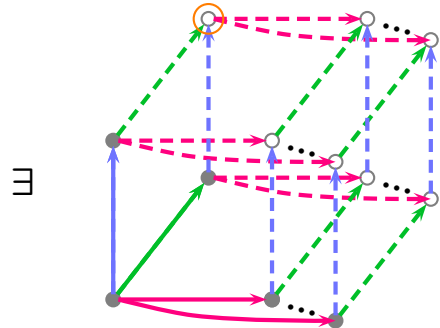
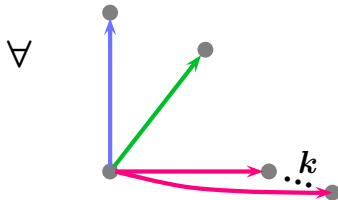


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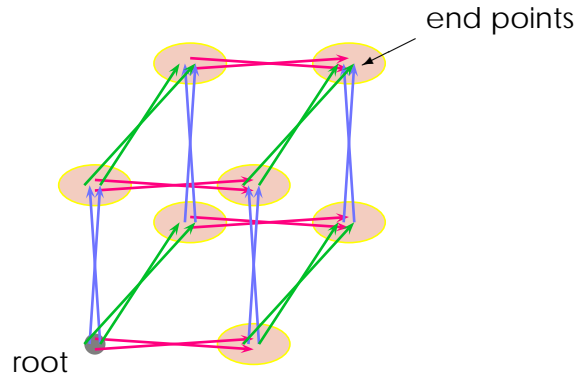


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- \mathfrak{F}_k is **finite** \rightsquigarrow 'atoms' of a model over \mathfrak{F}_k form a p-morphic image of \mathfrak{F}_k

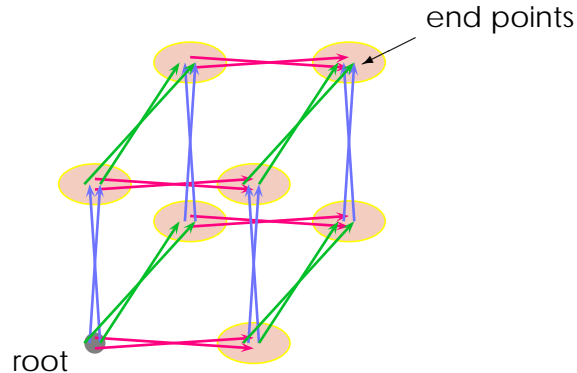
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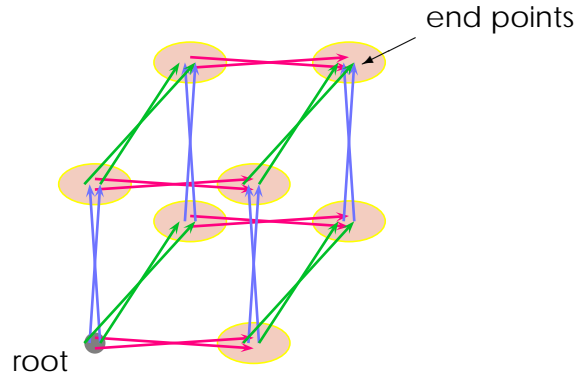
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- few propositional variables can partition end-points to few blocks only

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