An Extension of Kracht's Theorem to Generalized Sahlqvist Formulas

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$$\mathcal{M}l_{\Lambda}:$$

$$p_1, p_2, \cdots$$

$$\wedge, \vee, \neg, \longrightarrow$$

$$\top, \bot$$

$$\Diamond_\lambda, \Box_\lambda, \quad \lambda \in \Lambda$$

$$POS: \quad p_i, \top, \bot, \wedge, \vee, \Box_\lambda, \Diamond_\lambda$$

$$NEG: \quad \text{negation of } POS$$

Sahlqvist formula:

$$BA: \quad \Box_{\lambda_1} \ldots \Box_{\lambda_n} p_i$$

$$SA: \quad BA, NEG, \wedge, \vee, \Diamond_\lambda$$

$$SI: \quad SA \rightarrow POS$$

$$SF: \quad SI, \Box_\lambda, \wedge, \vee^*$$

Theorem. Every Sahlqvist formula has a computable local first-order equivalent.

Kracht formulas

$$FO_{\Lambda}:$$

$$R_{\lambda}, \quad \lambda \in \Lambda$$

Restricted quantification:

$$\forall y(xR_\lambda y \rightarrow \alpha(y)) \qquad (\forall y \triangleright_\lambda x)\alpha(y)$$

$$\exists y(xR_\lambda y \wedge \alpha(y)) \qquad (\exists y \triangleright_\lambda x)\alpha(x)$$

$$\Delta = \Box_{\lambda_1} \dots \Box_{\lambda_n}$$

We extend the set of predicate symbols with R_Δ for any Δ, and will allow the atomic formulas $xR_\Delta y$.

($xR_\Delta y$ iff there is a sequence of points $v_0, \dots, v_n$ such that $v_0 = x, v_n = y$ and $x_{i-1}R_{\lambda_i}x_i$ for $1 \leq i \leq n$)

We call a formula *restrictedly positive* if it is built up from atomic formulas, using $\wedge$, $\vee$ and restricted quantifiers only.

A formula is called *clean* if any variable is quantified only once.

We say that an occurrence of the variable y in the clean formula α is *inherently universal* if either y is free, or else y is bound by a restricted quantifier of the form $(\forall y \triangleright x)\beta$ which is not in the scope of an existential quantifier.

$\alpha(x)$ is a *Kracht formula* if

- α is clean,
- α is restrictedly positive,
- every atomic formula is either of the form $u = u$ or $u \neq u$ or has a form $xR_{\Delta}y$ $(n \geq 0)$ where at least one variable of x and y is inherently universal.

$\alpha(x)$ is a *Kracht formula* if

- α is clean,
- α is restrictedly positive,
- every atomic formula is either of the form $u = u$ or $u \neq u$ or has a form $xR_{\triangle}y$ $(n \geq 0)$ where x is inherently universal.

Theorem (Kracht)

Claim 1. Every Sahlqvist formula locally corresponds to some Kracht formula.

Claim 2. Every Kracht formula locally corresponds to some Sahlqvist formula.

Generalized Sahlvist formulas

(Goranko, Vakarelov)

Boxed atom:

$$BA: \quad p_i \quad | \quad \Box_\lambda BA$$

Box-formula

$$BF: \quad p_i \quad | \quad \Box_\lambda BA \quad | \quad POS \rightarrow BF$$

Examples of box-formulas

$$\Box_{\lambda_1} \ldots \Box_{\lambda_n} p_j$$

$$\Box_{\lambda_1}(POS_1 \rightarrow \Box_{\lambda_2}(POS_2 \rightarrow p_j))$$

p_j is a head

Let A be a set of box-formulas.

Dependency graph of A:

$$G = (V_A, E_A)$$

V_A contains all variables which occur in A

$p_i E_A p_j \iff p_i$ occurs in a formula from A

with a head p_j

A is *inductive* if G_A is acyclic

Generalized Sahlqvist formula:

$$BF: \quad p_i \quad | \quad \Box_\lambda BF \quad | \quad POS \to BF$$

$$GSA: \quad BF, NEG, \wedge, \vee, \Diamond_\lambda, \quad BF(GSA) \text{ is inductive}$$

$$GSI: \quad GSA \to POS$$

$$GSF: \quad GSI, \Box_\lambda, \wedge, \vee^*$$

Theorem Every generalized Sahlqvist formula has a computable local first-order equivalent.

What first-order formulas we obtain?

$$\Box_{\lambda_1} \ldots \Box_{\lambda_n} p_j \qquad R_{\lambda_1} \ldots R_{\lambda_n}$$

$$BF \qquad ?$$

$$L : x_i, \cap, \cup, R_\lambda^{-1}, R_\lambda^\Box, R_\lambda, \top, \bot.$$

Here $\bot, \top, x_i$ are atoms, $R_\lambda^{-1}, R_\lambda^\Box, R_\lambda$ are unary connectives, $\cap, \cup$ are binary connectives.

(W, R_λ, x_i) is a model with universe W, binary predicates R_λ and constants x_i

$$x_i = \{x_i\}$$

$$\top = W$$

$$\bot = \emptyset$$

$$R_\lambda^{-1}(A) = \{x | \exists y \in A x R_\lambda y\}$$

$$R_\lambda^{\Box}(A) = \{x | \forall y (x R_\lambda y \rightarrow y \in A)\}$$

$$R_\lambda(A) = \{x | \exists y \in A y R_\lambda x\}$$

Let $\mathcal{K}$ be the minimal class of expressions satisfying the conditions:

- $\{x_1, \ldots, x_n\} \subseteq \mathcal{K}$;

- if $S \in \mathcal{K}$, then $R_\lambda(S) \in \mathcal{K}$;

- if $B \subseteq \mathcal{K}$ and $S \in \mathcal{K}$ then $S \cap POS(B) \in \mathcal{K}$

$$(POS(B): \quad B, \cap, \cup, R_\lambda^{-1}, R_\lambda^{\Box}, \top, \bot)$$

Let $\phi \in L$ and $\psi \in \mathsf{Sub}(\phi)$

We say that a subexpression ψ is *safe* if one of the following holds:

1) $\psi = x_i$;

2) $\psi = R_\lambda(\psi')$, where ψ' is safe;

3) $\psi = \psi' \cap \psi''$, where either ψ' or ψ'' is safe.

We say that an expression ϕ is safe if

1)ϕ is safe as a subexpression of itself;

2)for every $R_\lambda(\psi) \in Sub(\phi)$ the subexpression ψ is safe.

Claim:

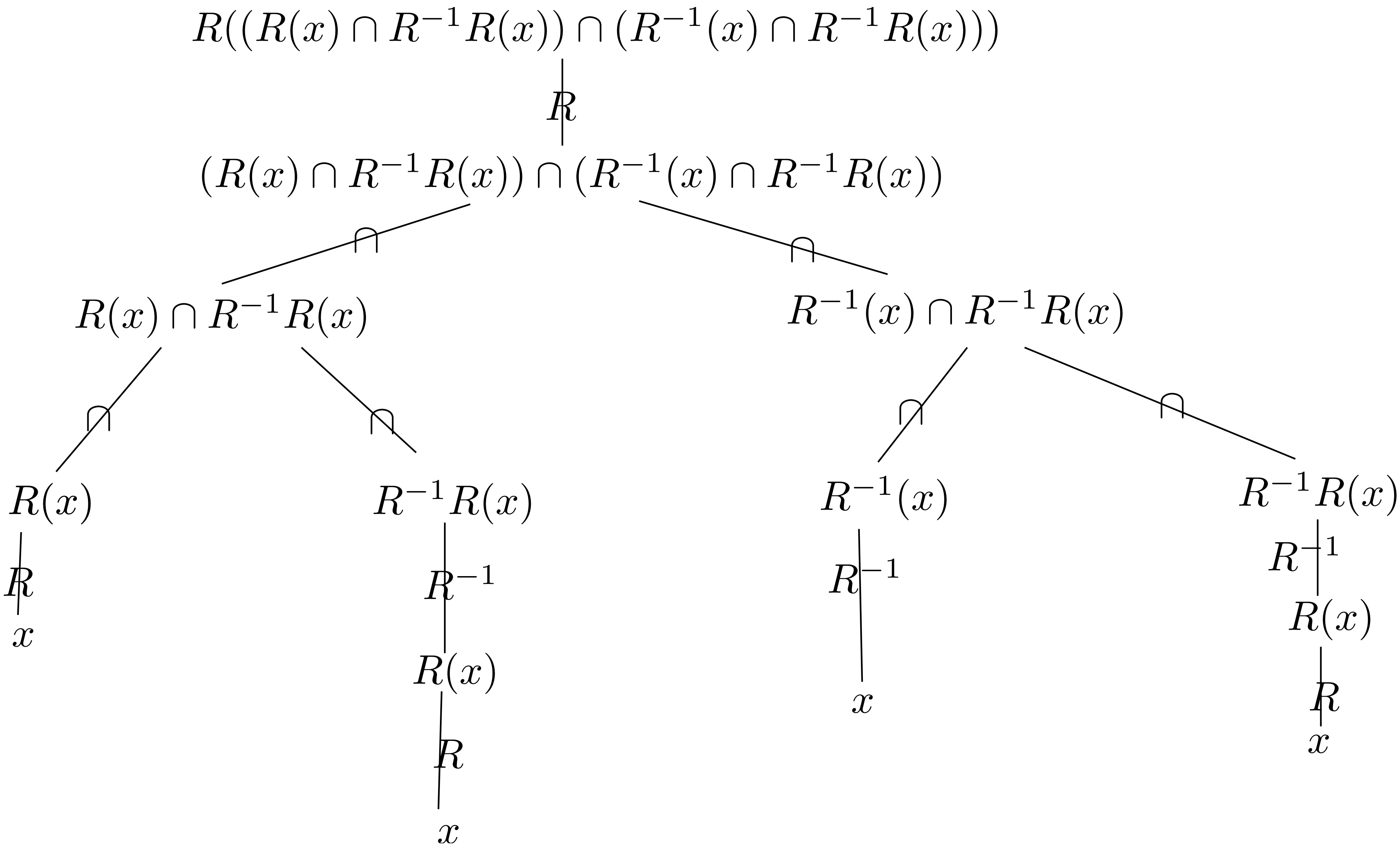
$$\mathcal{K} = \{S \in L | S \text{ is safe } \}$$

Examples of safe expressions

$$x_i, R(x), R(R(x) \cap R^{-1}R(x))$$

$$R\left(\left(R(x) \cap R^{-1}R(x)\right) \cap \left(R^{-1}(x) \cap R^{-1}(R(x))\right)\right)$$

There is a lineal algorithm, which takes an expression $\phi \in L$ and determines whether ϕ is safe.



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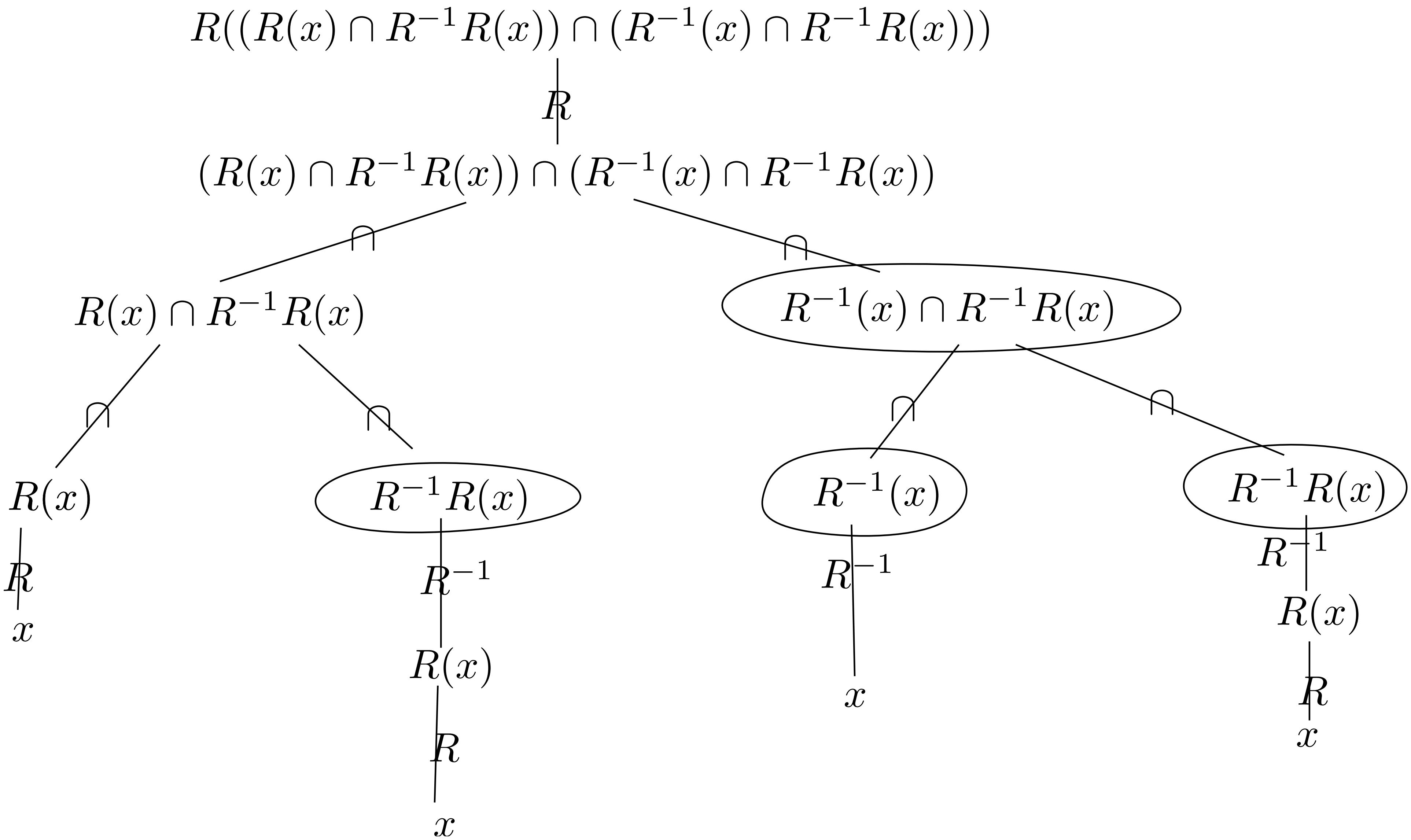

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Generalized Kracht's formulas

$\Box_{\lambda_1} \ldots \Box_{\lambda_n} p_j \qquad R_{\lambda_1} \ldots R_{\lambda_n}$

BF Safe expressions

For any safe expression $S(x_1, \ldots, x_n)$ we add to our signature the predicate $y \in S(x_1, \ldots, x_n)$.

$\alpha(x)$ is a *Generalized Kracht formula* if

- α is clean,
- α is restrictedly positive,
- every atomic formula is either of the form $u = u$ or $u \neq u$ or has a form $x \in S(x_1, \ldots, x_n)$, where S is a safe expression and $x_1, \ldots, x_n$ are inherently universal.

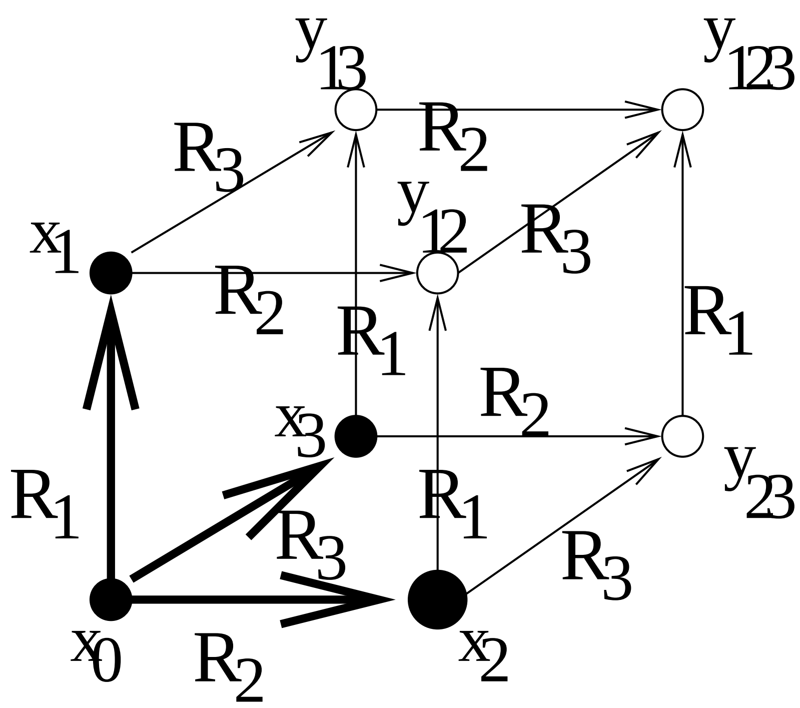
Theorem

Claim 1. Every generalized Sahlqvist formula locally corresponds to some generalized Kracht formula.

Claim 2. Every generalized Kracht formula locally corresponds to some generalized Sahlqvist formula.

Example 1. The formula cub_1 is theorem of $\mathbf{K}^3$ (V. Shehtman, 1978):

$$cub_1 = [\Diamond_1(\Box_2 p_{12} \wedge \Box_3 p_{13}) \wedge \Diamond_2(\Box_1 p_{21} \wedge \Box_3 p_{23}) \wedge \Diamond_3(\Box_1 p_{31} \wedge \Box_2 p_{32}) \wedge$$
$$\Box_1\Box_2(p_{12} \wedge p_{21} \to \Box_3 q_3) \wedge \Box_1\Box_3(p_{13} \wedge p_{31} \to \Box_2 q_2) \wedge \Box_2\Box_3(p_{23} \wedge p_{32} \to \Box_1 q_1)\,]$$
$$\to \Diamond_1\Diamond_2\Diamond_3(q_1 \wedge q_2 \wedge q_3).$$



y13
y123
R3
R2
y12
x1
R3
R2
R1
R1
R2
x3
y23
R1
R1
R3
R3
x0
R2
x2

$$\forall x_1 \rhd_1 x \forall x_2 \rhd_2 x \forall x_3 \rhd_3 x \exists y ((x R_1 R_2 R_3 y) \wedge$$

$$\wedge y \in R_3(R_2(x_1) \cap R_1(x_2)) \wedge y \in R_2(R_3(x_1) \cap R_1(x_3)) \wedge$$

$$\wedge y \in R_1(R_2(x_3) \cap R_3(x_2))).$$

Example 2. (Goranko, Vakarelov)

$$D_2 = p \wedge \Box(\Diamond p \to \Box q) \to \Diamond\Box\Box q$$

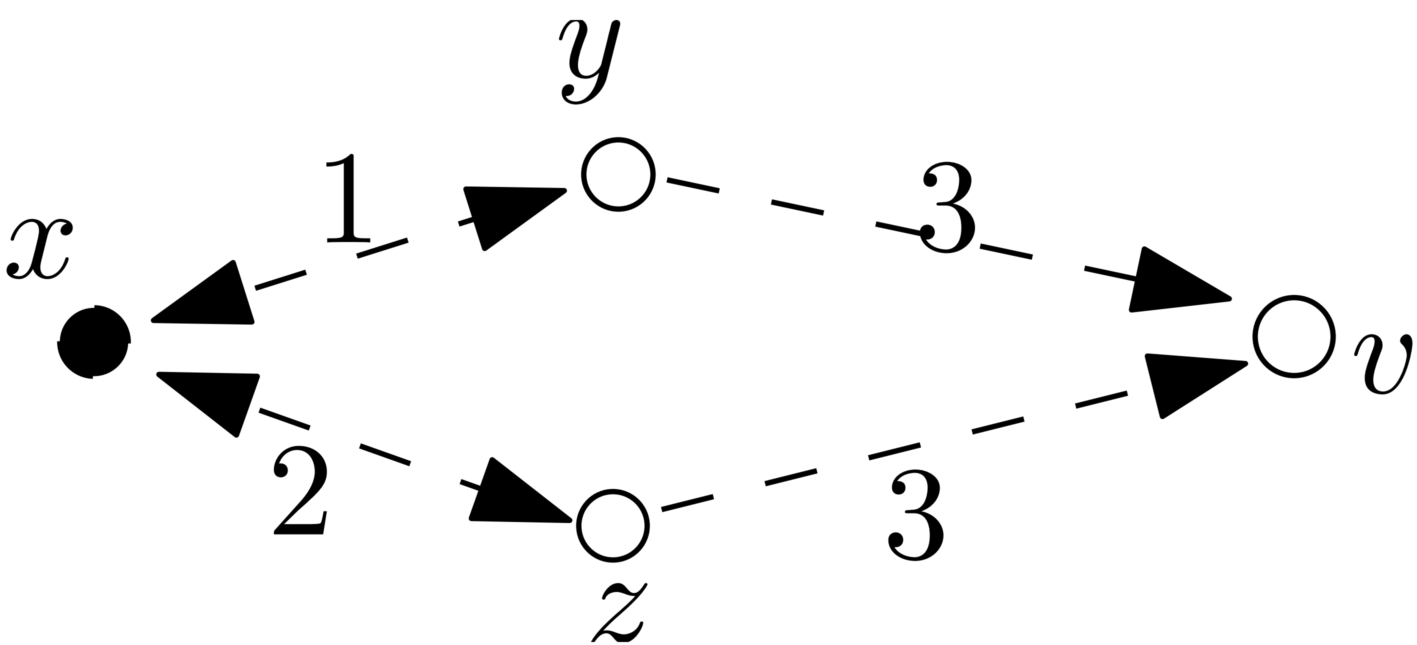
Its first-order correspondent is

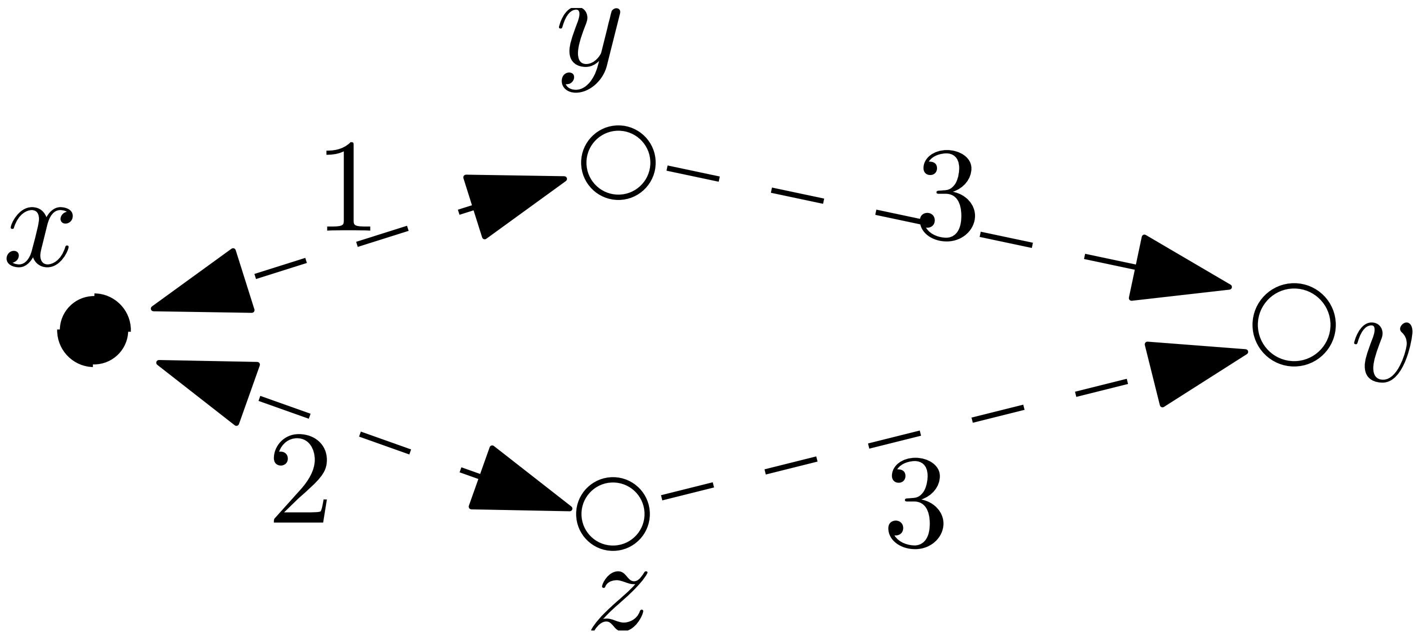
$$\exists y \left(xRy \wedge \forall z \left(yR^2 z \to z \in R(R(x) \cap R^{-1}(x)) \right) \right).$$

Example 3.

$$p \wedge \Box_1(\Diamond_1 p \to \Box_3 r) \to \Diamond_2(\Diamond_2 p \wedge \Diamond_3 r)$$

$$\exists y \exists z \exists v (xR_1y \wedge yR_1x \wedge xR_2z \wedge zR_2x \wedge yR_3v \wedge zR_3v)$$





$$\exists y \left(xR_1y \wedge yR_1x \wedge \right.$$

$$\left. \wedge \exists v \left(yR_3v \wedge v \in R_3(R_2(x) \cap R_2^{-1}(x)) \right) \right)$$