Properties of logics of individual and group agency

Andreas Herzig and Francois Schwarzentruber (IRIT, Toulouse, France)

This work is a link between:

- the STIT community;
- the many-dimensional modal logics community.

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2 Recalling $S5^n$

3 Results about group STIT

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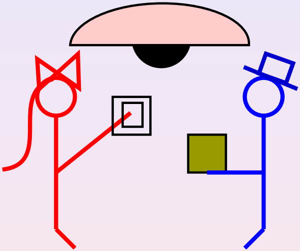
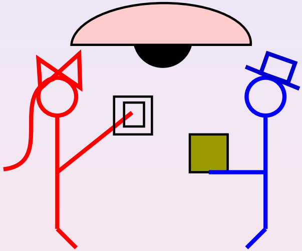
[Belnap and al. Facing the Future.]
[Nicolas Troquard. Independent agents in branching time: Towards a unified framework for reasoning about multiagent systems. PhD thesis.]

- Here, we only deal with Chellas' STIT.



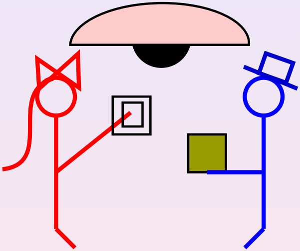
[*Barbara*]*light*

"*Barbara* sees to it that the *light* will be on (whatever the other agents do)".



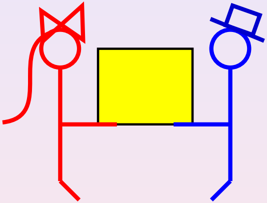
$[Pablo]cup\_empty$

"Pablo ensures his cup to be empty (whatever the other agents do)".



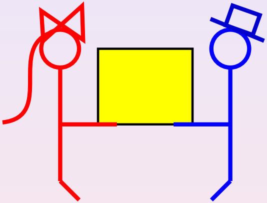
$\langle Pablo \rangle light$

"Pablo allows the light to be on."



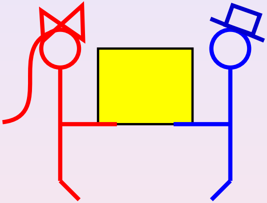
$\neg[Pablo]box\_in\_the\_kitchen$

"Pablo does not ensure the box to be in the kitchen."



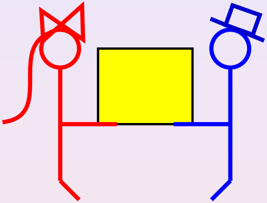
$$[\{Barbara, Pablo\}]box\_in\_the\_kitchen$$

"Barbara and Pablo ensure the box to be in the kitchen."



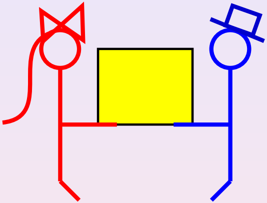
$[\emptyset] box\_is\_yellow$

"The nature (nobody) ensure the box to be yellow (i.e. the box is necessarily yellow)"



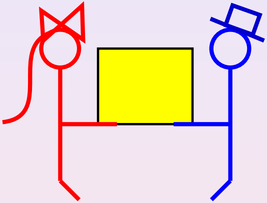
$\langle\emptyset\rangle box\_in\_the\_kitchen$

“The nature allows the box to be in the kitchen (i.e. it is possible that box will be in the kitchen.)”



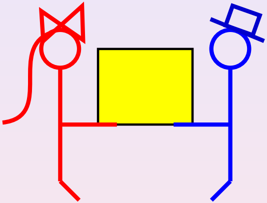
$$[\{Barbara, Pablo\}]box\_in\_the\_kitchen \wedge$$
$$\neg[\{Barbara\}]box\_in\_the\_kitchen \wedge$$
$$\neg[\{Pablo\}]box\_in\_the\_kitchen$$

"Barbara and Pablo are responsible for the box being in the kitchen."



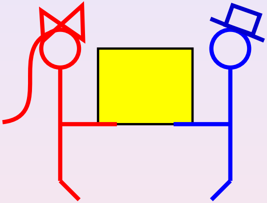
$[Barbara]box\_is\_yellow$

"Barbara ensures the box to be yellow (because the box is necessarily yellow)"



$\neg([Barbara]box\_is\_yellow \wedge \neg[\emptyset]box\_is\_yellow)$

"Barbara is not responsible for the box being yellow."



$\langle \emptyset \rangle [Barbara] \neg box\_in\_the\_kitchen$

"It is possible that Barbara ensures the box not to be in the kitchen. (i.e. Barbara can see to it that the box is not in the kitchen.)"

Outline

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   - Introduction
   - Syntax
   - Semantics
2. Recalling $S5^n$
3. Results about group STIT
4. Fragments
5. Conclusion

Syntax

Definition

Language $\mathcal{L}_{group\text{STIT}}$:

$$\varphi ::= p \mid \varphi \wedge \varphi \mid \neg\varphi \mid [J]\varphi$$

where $J \subseteq AGT$.

As usual, $\langle J\rangle\varphi \overset{\text{def}}{=} \neg[J]\neg\varphi$.

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Semantics

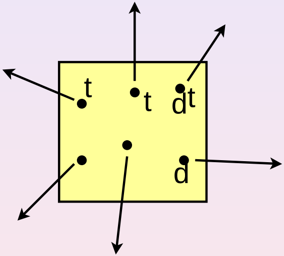


Figure: a choice cell

A choice cell with choices of agent Barbara

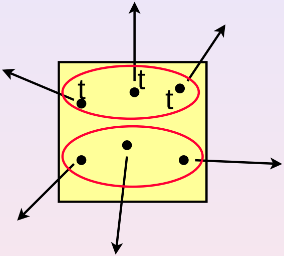


Figure: A cell of choices with $R_{Barbara}$ (equivalence relation for agent Barbara)

We group all states where Barbara is going to perform the same action in an equivalence class.

A choice cell with choices of agent Pablo

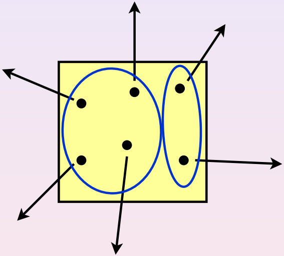


Figure: A choice cell with R_{Pablo} (equivalence relation for agent Pablo)

Is this a good choice cell?

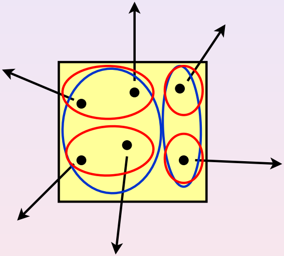
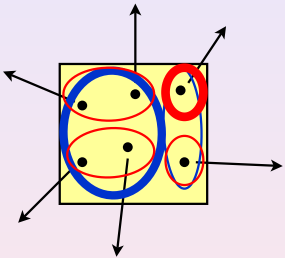


Figure: A choice cell with $R_{Barbara}$ and R_{Pablo}?

No : Inconsistent choices of agents are disallowed.



We need the property of *independence of agents*:

$$\text{for all } (x_a)_{a \in AGT} \in W^{AGT}, \bigcap_{a \in AGT} R_a(x_a) \neq \emptyset$$

Example

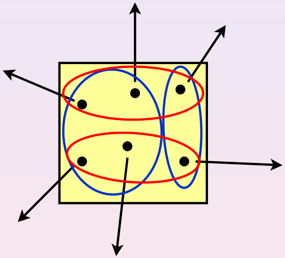


Figure: A correct choice cell with $R_{Barbara}$ and R_{Pablo}

What is a group choice?

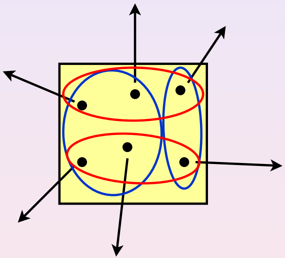


Figure: What is $R_{\{Barbara,Pablo\}}$?

Group choice

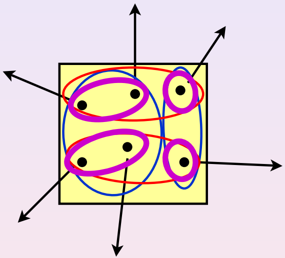


Figure: Take the intersection (Horty's idea)!

$$R_G = \bigcap_{a \in G} R_{\{a\}}$$

*group*STIT-model

Definition

A *group*STIT-model is a tuple $\mathcal{M} = \langle W, R, V \rangle$ such that:

- W is a non-empty set;
- $R : 2^{AGT} \to 2^{W \times W}$ such that:
  - $R_\emptyset = W \times W$;
  - for all $G \subseteq AGT$, $R_G = \bigcap_{a \in G} R_{\{a\}}$;
  - for all $(x_a)_{a \in AGT} \in W^{AGT}$, $\bigcap_{a \in AGT} R_a(x_a) \neq \emptyset$.
- $V : W \to 2^{ATM}$.

Semantics

Definition

$\mathcal{M}, w \models [J]\varphi$ iff for all $w' \in R_J(w)$, $\mathcal{M}, w' \models \varphi$.

Outline

1. Recalling group STIT
2. Recalling $S5^n$
   - Definition
   - Results
3. Results about group STIT
4. Fragments
5. Conclusion

[D.M. Gabbay, A. Kurucz, F. Wolter, M. Zakharyaschev.
Many-dimensional modal logics: Theory and Applications.]

Syntax of $S5^n$

Definition

Language $\mathcal{L}_{S5^n}$:

$$\varphi \ ::= \ p \mid \neg\varphi \mid (\varphi \land \varphi) \mid [\bar{i}]\varphi$$

where $i \in AGT$.

Models of $S5^n$

Definition ($S5^n$ model)

A $S5^n$-model is a tuple $\mathcal{X} = (X, R, V)$ where:

- $X = X_1 \times \ldots \times X_n$;
- $R_{\bar{i}} = \{\langle(x_1, \ldots, x_n), (y_1, \ldots, y_n)\rangle \in X^2 \mid x_j = y_j \text{ for all } j \neq i\}$
- $V : ATM \to 2^X$.

Semantics

Definition

$\mathcal{M}, w \models [\bar{i}]\varphi$ iff for all $w' \in R_{\bar{i}}(w)$, $\mathcal{M}, w' \models \varphi$.

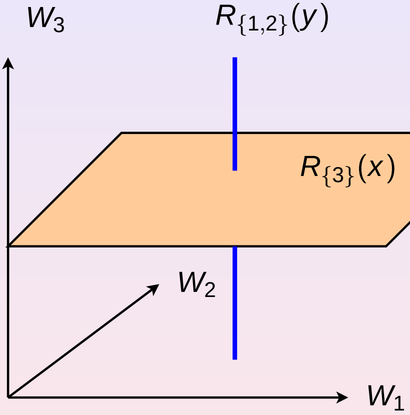


Figure: Example of a $S5^n$-model.

In $S5^n$, in which we can say for example:
$[\overline{\{3\}}][\overline{\{4\}}]p$
"At the point x, p is true in every point of the subspace of dimension 2 parallel to axis 3 and 4."

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What we know about $S5^n$

Theorem

The problem of satisfiability of $S5^n$ is:

- *NP-complete if $n = 1$;*
- *NEXPTIME-complete if $n = 2$.*
- *undecidable if $n \geq 3$.*

[D.M. Gabbay, A. Kurucz, F. Wolter, M. Zakharyaschev. Many-dimensional modal logics: Theory and Applications.]

What we know about $S5^n$

Theorem

The logic $S5^n$ is:

- *axiomatizable if $n \leq 2$;*
- *non-axiomatizable if $n \geq 3$.*

[D.M. Gabbay, A. Kurucz, F. Wolter, M. Zakharyaschev. Many-dimensional modal logics: Theory and Applications.]

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     - Satisfiability
   - Satisfiability problem in group STIT
   - Axiomatizability of group STIT
4. Fragments

Proposition

Let $J_1, J_2 \subseteq AGT$. We have:

$$\models_{groupSTIT} [J_1 \cap J_2]\varphi \leftrightarrow [J_1][J_2]\varphi$$

.

Proof.

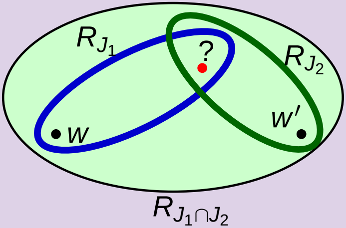
$\models_{groupSTIT} [J_1 \cap J_2]\varphi \rightarrow [J_1][J_2]\varphi$

$R_{J_1} \circ R_{J_2} \subseteq R_{J_1 \cap J_2} \circ R_{J_1 \cap J_2} \subseteq R_{J_1 \cap J_2}$.

□

Proof.

$\models_{groupSTIT} [J_1][J_2]\varphi \rightarrow [J_1 \cap J_2]\varphi$ Let w, w' be such that $wR_{J_1 \cap J_2}w'$.



Here, we need the *independence of agents*. $R_{J_1}(w) \cap R_{J_2}(w') = R_{J_1 \cap J_2}(w) \cap R_{J_1 \setminus J_1 \cap J_2}(w) \cap R_{J_1 \cap J_2}(w') \cap R_{J_2 \setminus J_1 \cap J_2}(w') \neq \emptyset$.

□

So any group STIT formula can be written with $[\bar{i}]$ and $[AGT]$ operators.

Example

if $AGT = \{1, 2, 3, 4, 5\}$,

$$[AGT]p \wedge [2,3]q \equiv [AGT]p \wedge [\bar{1}][\bar{4}][\bar{5}]q$$

.

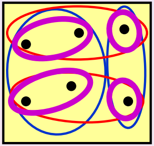


Figure: A STIT-model

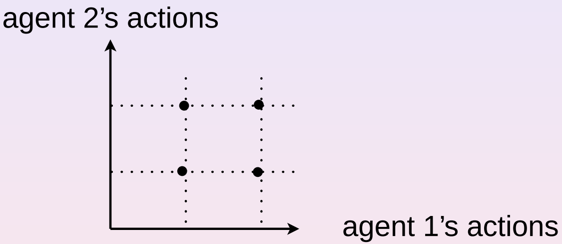


Figure: A $S5^n$-model

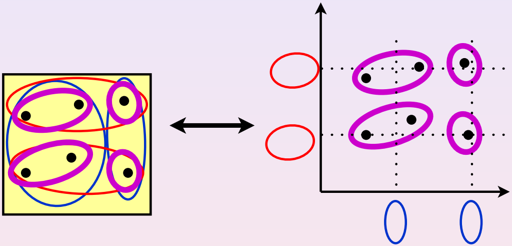


Figure: A STIT-model and an $S5^n$-model

Theorem

For every $\varphi \in \mathcal{L}_{S5^n}$, the following are equivalent:

1. *φ is $S5^n$-satisfiable;*
2. *φ is satisfiable in a groupSTIT model where $R_{AGT} = id_W$;*
3. *$[\emptyset] \bigwedge_{p \in atm(\varphi)}([AGT]p \leftrightarrow p) \wedge \varphi$ is groupSTIT-satisfiable.*

Proof.

φ is $S5^n$-satisfiable
$\Rightarrow \varphi$ is satisfiable in a *group*STIT model where $R_{AGT} = id_W$

because a $S5^n$-model can be transformed in a *group*STIT model where $R_{AGT} = id_W$. □

Proof.

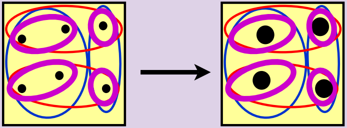
φ is satisfiable in a *group*STIT model where $R_{AGT} = id_W$
$\Rightarrow [\emptyset] \bigwedge_{p \in atm(\varphi)} ([AGT]p \leftrightarrow p) \wedge \varphi$ is *group*STIT-satisfiable

ok!

□

Proof.

$[\emptyset] \bigwedge_{p \in atm(\varphi)}([AGT]p \leftrightarrow p) \wedge \varphi$ is *group*STIT-satisfiable
$\Rightarrow \varphi$ is satisfiable in a *group*STIT model where $R_{AGT} = id_W$

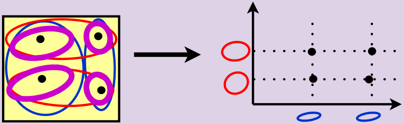


From the model $\mathcal{M}' = (W', R', V')$ satisfying $[\emptyset] \bigwedge_{p \in atm(\varphi)}([AGT]p \leftrightarrow p) \wedge \varphi$ we construct a new model $\mathcal{M} = (W, R, V)$ for φ.

- $W = \{R'_{AGT}(x) \mid x \in W'\}$;
- $V(p) = \{R'_{AGT}(x) \in W \mid R'_{AGT}(x) \subseteq V'(p)\}$.

Proof.

φ is satisfiable in a *group*STIT model where $R_{AGT} = id_W$
$\to \varphi$ is S5^n-satisfiable



From the *group*STIT model $\mathcal{M} = (W, R, V)$ satisfying φ we construct an S5^n model $\mathcal{X}' = (X', R', V')$ for φ.

- $X' = X_1 \times \ldots \times X_n$ where for all $i \in AGT$, $X_i = \{R_i(w) \mid w \in W\}$;
- $V'(p) = \{(x_1, \ldots, x_n) \mid \bigcap_{i \in AGT} x_i \in V(p)\}$

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     - Models
     - Satisfiability
   - Satisfiability problem in group STIT
   - Axiomatizability of group STIT
4. Fragments

Theorem

The satisfiability problem in groupSTIT is:

- *NP-complete if* $card(AGT) = 1$;
- *NEXPTIME-complete if* $card(AGT) = 2$.
- *undecidable if* $card(AGT) \geq 3$.

Theorem

The satisfiability problem in groupSTIT is:

- *NP-complete if* $card(AGT) = 1$.
  We have two operators : $[\emptyset]$ *and* $[\{1\}]$ *where* $\models [\emptyset]\varphi \to [\{1\}]\varphi$. *We use a selection-of-points argument;*
- *NEXPTIME-complete if* $card(AGT) = 2$.
- *undecidable if* $card(AGT) \geq 3$.

Theorem

The satisfiability problem in groupSTIT is:

- *NP-complete if* $card(AGT) = 1$;
- *NEXPTIME-complete if* $card(AGT) = 2$.
  proof: by filtration.
- *undecidable if* $card(AGT) \geq 3$.

[Balbiani, Herzig, Troquard. JPL 2008.]

Theorem

The satisfiability problem in groupSTIT is:

- *NP-complete if* $card(AGT) = 1$.
- *NEXPTIME-complete if* $card(AGT) = 2$.
- *undecidable if* $card(AGT) \geq 3$.
  proof: reduction from $S5^n$ *to groupSTIT using the translation from* $S5^n$ *to groupSTIT.*

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Theorem

The logic groupSTIT is axiomatizable if $card(AGT) \leq 2$.

[Balbiani, Herzig, Troquard. JPL 2008.]

Theorem

The logic groupSTIT is not axiomatizable if $card(AGT) \geq 3$.

Proof that *group*STIT is not axiomatizable if $card(AGT) \geq 3$

- Suppose *group*STIT has an axiomatics Ax;
- Let us denote Ax' obtained by Ax by removing $[AGT]$-operators;

Example

$(card(AGT) = 4)$
If $\psi = [2,4]\varphi \rightarrow [AGT]\varphi \in Ax$, then $\psi' = [\overline{1}][\overline{3}]\varphi \rightarrow \varphi \in Ax'$.

- Then we prove that $\models_{S5^n} \varphi$ iff $\vdash_{Ax'} \varphi$.

$\vdash_{Ax'} \varphi \rightarrow \models_{S5^n} \varphi$

Prove it for a axiom $\psi' \in Ax'$.

- $\psi' \in Ax'$ comes from $\psi \in Ax$;
- ψ is valid in *group*STIT models;
- ψ is valid in *group*STIT models where $R_{AGT} = id_W$;
- ψ' is valid in *group*STIT models where $R_{AGT} = id_W$;
- ψ' is valid in $S5^n$ models.

$$\boxed{\models_{S5^n} \varphi \rightarrow \vdash_{Ax'} \varphi}$$

$\models_{S5^n} \varphi$	iff	$\models_{group\text{STIT}} [\emptyset] \bigwedge_{p \in atm(\varphi)} ([AGT]p \leftrightarrow p) \rightarrow \varphi$
	iff	$\vdash_{Ax} [\emptyset](\bigwedge_{p \in atm(\varphi)} [AGT]p \leftrightarrow p) \rightarrow \varphi$
	implies	$\vdash_{Ax, [AGT]\psi \leftrightarrow \psi} \varphi$
	implies	$\vdash_{Ax'} \varphi$

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   - Restricting coalitions
     - To singletons: individualSTIT
     - To totally ordered coalitions
   - Restricting expressivity: “can do”
     - Definition
     - Problem of satisfiability

Definition

Language $\mathcal{L}_{individual\mathrm{STIT}}$: fragment of *group*STIT

$$\varphi ::= p \mid \varphi \wedge \varphi \mid \neg\varphi \mid [\{a\}]\varphi \mid [\emptyset]\varphi$$

where $a \in AGT$.

Example

$[\{Barbara, Pablo\}]light$ is not in $\mathcal{L}_{individual}$STIT.

Example

$[\{Barbara\}]light \vee \langle\emptyset\rangle\neg light$ is in $\mathcal{L}_{individual}$STIT.

individual STIT is close to a logic in which we can say:

$[2]p$

“At the point x, p is true in every point of the *hyperplane* orthogonal to axis 2.”

$[\emptyset]p$

“Everywhere, p is true.”

Theorem

The problem of satisfiability of individualSTIT is:

- *NP-complete if* $card(AGT) = 1$;
- *NEXPTIME-complete if* $card(AGT) \geq 2$.

[Balbiani, Herzig, Troquard. JPL 2008.]

Theorem

If we restrict to nested coalitions like $\{1\}$, $\{1,2\}$,$\{1,2,3\}$ *etc., then the problem of satisfiability is NP-complete.*

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   - Restricting expressivity: “can do”
     - Definition
     - Problem of satisfiability

Definition

Language $\mathcal{L}_{group\mathrm{STIT}_{cando}}$: fragment of *group*STIT

$$\varphi ::= p \mid \varphi \land \varphi \mid \neg\varphi \mid \langle\emptyset\rangle[J]\varphi$$

where $J \subseteq AGT$.

Example

$\langle\emptyset\rangle[\{Barbara, Pablo\}]light \wedge \langle\emptyset\rangle[\{Barbara\}]\neg light$ is in $\mathcal{L}_{group\mathrm{STIT}_{cando}}$.

Example

$[\{Barbara, Pablo\}]light \wedge \langle\emptyset\rangle[\{Barbara\}]\neg light$ is <u>not</u> in $\mathcal{L}_{group\mathrm{STIT}_{cando}}$.

- In *group*STIT$_{cando}$, we can only say: $\langle\emptyset\rangle[3]p$
  “There is a subspace of dimension 3 parallel to axes 1, 2, and 4 in which p is true”.

Theorem

The problem of satisfiability of $\mathcal{L}_{groupSTIT_{cando}}$ is NP-complete. (whatever the number of agents).

In this paper we have established the link between STIT and product logics.
Future research:

- Decidable and axiomatizable fragments of group STIT (or $S5^n$) in which we can deal with responsability etc.
- Logic with a restriction on the number of actions [A. Herzig and E. Lorini. A dynamic logic of agency. Technical report, Institut de Recherche en Informatique de Toulouse (IRIT), Toulouse, France, 2008.];
- Combining with time, knowledge etc.
- Comparison with extensions of STIT (achievement STIT).

Thank you!