Many-Valued Hybrid Logic

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Advances in Modal Logic
10th of September 2008
Nancy, France

Outline

Introduction and motivation

In the two-valued case **hybrid logics** are extensions of classical modal logics that adds:

- Nominals as names of worlds.
- Satisfaction operators (can express facts at particular worlds).
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These extensions have several characteristics, for instance:

- They increase the expressive power of the language.
- They normally result in nice proof theory.

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- How should such extensions look?
- Do they have the same nice properties as in the classical two-valued setting?

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Partial answer: *In this paper we present a many-valued hybrid logic that seems natural and have a well-behaved proof theory in form of straightforward tableau system. (Thus in a sense hybrid logic extensions do not depend in any essential way on the underlying two-valued logic.)*

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Partial answer: *In this paper we present a many-valued hybrid logic that seems natural and have a well-behaved proof theory in form of straightforward tableau system. (Thus in a sense hybrid logic extensions do not depend in any essential way on the underlying two-valued logic.) Furthermore this allows us to show the decidability of the logic using known techniques from hybrid logic.*

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This many-valued modal logic have the following features, which we adopt:

- The truth values constitute a finite Heyting algebra.
- The accessibility relation is allowed to be many-valued as well.

(A Heyting algebra is a lattice with an additional operation called pseudo complement. The pseudo complement of a relative to b, denoted $a \Rightarrow b$, is defined to be the greatest element x of $\mathcal{T}$ such that $a \sqcap x \leq b$.)

Syntax for MVHL

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The set of MVHL-formulas is then given by the following grammar:

$$\varphi \quad ::= \quad p \mid a \mid i \mid (\varphi \wedge \psi) \mid (\varphi \vee \psi) \mid (\varphi \rightarrow \psi) \mid \Box\varphi \mid \Diamond\varphi \mid @_i\varphi \mid E\varphi \mid A\varphi \ ,$$

where $p \in \text{PROP}$, $a \in \mathcal{T}$, and $i \in \text{NOM}$.

(A form of negation can be defined by $\neg\varphi := \varphi \rightarrow \bot$, where $\bot$ is the least element of $\mathcal{T}$.)

Semantics for MVHL

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A model $\mathcal{M}$ is a tuple $\mathcal{M} = \langle W, R, \mathbf{n}, \nu \rangle$, where:

- W is the set of worlds.
- R is a mapping $R : W \times W \to \mathcal{T}$, called the accessibility relation.
- $\mathbf{n}$ is a function interpreting the nominals, i.e. $\mathbf{n} : \mathrm{NOM} \to W$.
- ν is a valuation $\nu : W \times \mathrm{PROP} \to \mathcal{T}$ that assigns truth values to the propositional variables at each world.

Given a model $\mathcal{M} = \langle W, R, \mathbf{n}, \nu \rangle$, the valuation ν can be extended to the set of all MVHL-formulas in the following way:

$$
\begin{array}{rcl}
\nu(w, a) & := & a \quad \text{for } a \in \mathcal{T} \\
\nu(w, i) & := & \begin{cases} \top & \text{, if } \mathbf{n}(i) = w \\ \bot & \text{, else} \end{cases} \\
\nu(w, \varphi \wedge \psi) & := & \nu(w, \varphi) \sqcap \nu(w, \psi) \\
\nu(w, \varphi \vee \psi) & := & \nu(w, \varphi) \sqcup \nu(w, \psi) \\
\nu(w, \varphi \rightarrow \psi) & := & \nu(w, \varphi) \Rightarrow \nu(w, \psi) \\
\nu(w, \Box\varphi) & := & \bigsqcap \{R(w, v) \Rightarrow \nu(v, \varphi) \mid v \in W\} \\
\nu(w, \Diamond\varphi) & := & \bigsqcup \{R(w, v) \sqcap \nu(v, \varphi) \mid v \in W\} \\
\nu(w, @_i\varphi) & := & \nu(\mathbf{n}(i), \varphi) \\
\nu(w, A\varphi) & := & \bigsqcap \{\nu(v, \varphi) \mid v \in W\} \\
\nu(w, E\varphi) & := & \bigsqcup \{\nu(v, \varphi) \mid v \in W\}
\end{array}
$$

Some motivations for the choice of semantics

Some equivalences from classical Hybrid Logic that also holds in the many-valued setting:

$$\begin{aligned} \nu(w, @_i\varphi) &= \nu(w, E(i \wedge \varphi)) \\ \nu(w, @_i\varphi) &= \nu(w, A(i \rightarrow \varphi)) \end{aligned}$$

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$$\nu(w, @_i\Diamond j) = R(\mathbf{n}(i), \mathbf{n}(j)),$$

and so is equality of worlds:

$$\nu(w, @_i j) = \top \quad \text{iff} \quad \mathbf{n}(i) = \mathbf{n}(j).$$

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- Since we allow the accessibility relation to take many values as well, pseudo complements are also needed to interpret the modalities $\Box$ and $\Diamond$.
- The finiteness is essential for the formulation of the tableau rules and may also be for the decidability result.
- (The choice of a finite Heyting algebra follows naturally from Fittings motivation.)

A tableau system for MVHL

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The notions of tableaux are as usual.

Branch closing rules

A tableau branch Θ is said to be *closed* if one of the following holds:

1. $T@_i(a \rightarrow b) \in \Theta$, for some a, b with $a \not\leq b$.
2. $F@_i(a \rightarrow b) \in \Theta$, for some a, b with $a \leq b$, $a \neq \bot$, and $b \neq \top$.
3. $F@_i(\bot \rightarrow \varphi) \in \Theta$, for some formula φ.
4. $F@_i(\varphi \rightarrow \top) \in \Theta$, for some formula φ.
5. $T@_i(b \rightarrow \varphi), F@_i(a \rightarrow \varphi) \in \Theta$, for some a, b with $a \leq b$.

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5. $T@_i(b \longrightarrow \varphi), F@_i(a \longrightarrow \varphi) \in \Theta$, for some a, b with $a \leq b$.
6. $T@_j(a \longrightarrow i), F@_i(b \longrightarrow j) \in \Theta$, for some $a, b \neq \bot$.
7. $T@_i(i \longrightarrow a) \in \Theta$, for some nominal i and truth value a with $a \neq \top$.

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4. $F@_i(\varphi \rightarrow \top) \in \Theta$, for some formula φ.
5. $T@_i(b \rightarrow \varphi), F@_i(a \rightarrow \varphi) \in \Theta$, for some a, b with $a \leq b$.
6. $T@_j(a \rightarrow i), F@_i(b \rightarrow j) \in \Theta$, for some $a, b \neq \bot$.
7. $T@_i(i \rightarrow a) \in \Theta$, for some nominal i and truth value a with $a \neq \top$.

A *tableau proof* of a formula φ is a closed tableau with root $F@_i(\top \rightarrow \varphi)$, where i is an arbitrary nominal not occurring in φ. (A formula φ is said to be valid if for all models $\mathcal{M}$ and all worlds w in $\mathcal{M}$, $\nu(w, \varphi) = \top$.)

Propositional rules

$$\frac{T@_i(a \to (\varphi \land \psi))}{\begin{array}{c} T@_i(a \to \varphi) \\ T@_i(a \to \psi) \end{array}} \; (\mathbf{T}\land)^1 \qquad \frac{F@_i(a \to (\varphi \land \psi))}{F@_i(a \to \varphi) \mid F@_i(a \to \psi)} \; (\mathbf{F}\land)^1$$

$$\frac{T@_i((\varphi \lor \psi) \to a)}{\begin{array}{c} T@_i(\varphi \to a) \\ T@_i(\psi \to a) \end{array}} \; (\mathbf{T}\lor)^2 \qquad \frac{F@_i((\varphi \lor \psi) \to a)}{F@_i(\varphi \to a) \mid F@_i(\psi \to a)} \; (\mathbf{F}\lor)^2$$

$$\frac{F@_i(a \to (\varphi \to \psi))}{\begin{array}{c|c|c} T@_i(b_1 \to \varphi) & \cdots & T@_i(b_n \to \varphi) \\ F@_i(b_1 \to \psi) & \cdots & F@_i(b_n \to \psi) \end{array}} \; (\mathbf{F}\to)^3 \qquad \frac{T@_i(a \to (\varphi \to \psi))}{F@_i(b \to \varphi) \mid T@_i(b \to \psi)} \; (\mathbf{T}\to)^4$$

[1] Where $a \neq \bot$. [2] Where $a \neq \top$. [3] Where $a \neq \bot$ and $b_1, \ldots, b_n$ are all the members of $\mathcal{T}$ with $b_i \leq a$ except $\bot$. [4] Where $a \neq \bot$ and b is any member of $\mathcal{T}$ with $b \leq a$ except $\bot$.

Reversal rules

$$\frac{F@_i(a \to \varphi)}{T@_i(\varphi \to b_1) \mid \cdots \mid T@_i(\varphi \to b_n)}\ (\mathbf{F}\geq)^{1,2} \qquad \frac{T@_i(a \to \varphi)}{F@_i(\varphi \to b)}\ (\mathbf{T}\geq)^{1,3}$$

$$\frac{F@_i(\varphi \to a)}{T@_i(b_1 \to \varphi) \mid \cdots \mid T@_i(b_n \to \varphi)}\ (\mathbf{F}\leq)^{1,4} \qquad \frac{T@_i(\varphi \to a)}{F@_i(b \to \varphi)}\ (\mathbf{T}\leq)^{1,5}$$

[1] φ is a formula other than a propositional constant from $\mathcal{T}$. [2] Where $b_1, \ldots, b_n$ are all maximal members of $\mathcal{T}$ with $a \nleq b_i$ and $a \neq \bot$. [3] Where b is any maximal member of $\mathcal{T}$ with $a \nleq b$ and $a \neq \bot$. [4] Where $b_1, \ldots, b_n$ are all minimal members of $\mathcal{T}$ with $b_i \nleq a$ and $a \neq \top$. [5] Where b is any minimal member of $\mathcal{T}$ with $b \nleq a$ and $a \neq \top$.

Modal rules

$$\frac{F@_i(a \to \Box\varphi)}{\begin{array}{c|c|c} T@_i(b_1 \leftrightarrow \Diamond j) & \cdots & T@_i(b_n \leftrightarrow \Diamond j) \\ F@_j((a \sqcap b_1) \to \varphi) & \cdots & F@_j((a \sqcap b_n) \to \varphi) \end{array}} \ (\mathbf{F}\Box)^1$$

$$\frac{F@_i(\Diamond\varphi \to a)}{\begin{array}{c|c|c} T@_i(b_1 \leftrightarrow \Diamond j) & \cdots & T@_i(b_n \leftrightarrow \Diamond j) \\ F@_j(\varphi \to (b_1 \Rightarrow a)) & \cdots & F@_j(\varphi \to (b_n \Rightarrow a)) \end{array}} \ (\mathbf{F}\Diamond)^{1,2}$$

$$\frac{T@_i(a \to \Box\varphi) \qquad T@_i(b \to \Diamond j)}{T@_j((a \sqcap b) \to \varphi)} \ (\mathbf{T}\Box) \qquad \frac{T@_i(\Diamond\varphi \to a) \qquad T@_i(b \to \Diamond j)}{T@_j(\varphi \to (b \Rightarrow a))} \ (\mathbf{T}\Diamond)^2$$

[1] Where $\mathcal{T} = \{b_1, \ldots, b_n\}$ and j is a nominal new to the branch. [2] Where the principal premise is a quasi-subformula of the root formula.

Hybrid rules

$$\frac{T@_i(@_j\varphi \to a)}{T@_j(\varphi \to a)}\ (@_L) \qquad \frac{T@_i(a \to @_j\varphi)}{T@_j(a \to \varphi)}\ (@_R)$$

$$\frac{F@_i\varphi \quad T@_i(a \to j)}{F@_j\varphi}\ (\textbf{F-NOM})^{1,2} \qquad \frac{T@_i\varphi \quad T@_i(a \to j)}{T@_j\varphi}\ (\textbf{T-NOM})^{1,2}$$

$$\frac{T@_k(\Diamond i \to b) \quad T@_i(a \to j)}{T@_k(\Diamond j \to b)}\ (\textbf{BRIDGE}_L)^1 \qquad \frac{T@_k(b \to \Diamond i) \quad T@_i(a \to j)}{T@_k(b \to \Diamond j)}\ (\textbf{BRIDGE}_R)^1$$

$$\frac{T@_i(\top \to j) \quad T@_j(\top \to k)}{T@_i(\top \to k)}\ (\textbf{TRANS}) \qquad \frac{T@_i(a \to j)}{T@_i(\top \to j)}\ (\textbf{NOM EQ})^1$$

[1] Where $a \neq \bot$. [2] Where the principal premise is a quasi-subformula of the root formula.

Notes on the tableau system

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The rule **(T$\Box$)** is only non-trivial if a and b are not $\bot$. If we "read" $T@_i(\top \rightarrow \Box\varphi)$ as $@_i\Box\varphi$ we get the following reduction into the classical two-valued hybrid logic rule for $\Box$:

$$\frac{T@_i(\top \rightarrow \Box\varphi) \qquad T@_i(\top \rightarrow \Diamond j)}{T@_j((\top \sqcap \top) \rightarrow \varphi)} \quad \rightsquigarrow \quad \frac{@_i\Box\varphi \qquad @_i\Diamond j}{@_j\varphi}$$

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Similar for the other rules with some modifications.

Termination of the tableau system

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Note that it follows from our termination proof that also Fitting's original many-valued modal logic is decidable. This was not shown by Fitting.

Completeness of the tableau system

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Theorem (Completeness)

If there is no tableau proof of the formula φ, then there is a model $\mathcal{M} = \langle W, R, \mathbf{n}, \nu \rangle$ and a $w \in W$ such that $\nu(w, \varphi) \neq \top$.

Construction of the model

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A nominal i is called an *urfather* if $i = \mathbf{n}_\Theta(j)$ for some nominal j.

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Definition (Upper and lower bounds)

For a formula φ in the language of MVHL and a nominal i, define:

$$bound^{\Theta,i}(\varphi) = \bigsqcap\{a \mid T@_i(\varphi \to a) \in \Theta\}$$

$$bound_{\Theta,i}(\varphi) = \bigsqcup\{a \mid T@_i(a \to \varphi) \in \Theta\}$$

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Lemma (Lower bound less than upper bound)

For all i on Θ and all formulas φ of MVHL

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Lemma (Ordering lemma)

Let φ be any formula in the MVHL language other than a propositional constant from $\mathcal{T}$, and let $a \in \mathcal{T}$, then:

- *(i) If $T@_i(a \rightarrow \varphi) \in \Theta$, then $a \leq bound_{\Theta,i}(\varphi)$.*
- *(ii) If $T@_i(\varphi \rightarrow a) \in \Theta$, then $bound^{\Theta,i}(\varphi) \leq a$.*
- *(iii) If $F@_i(a \rightarrow \varphi) \in \Theta$, then $a \nleq bound^{\Theta,i}(\varphi)$.*
- *(iv) If $F@_i(\varphi \rightarrow a) \in \Theta$, then $bound_{\Theta,i}(\varphi) \nleq a$.*

Theorem (Main completeness theorem)

Let ν be a valuation such that for all propositional variables p and all urfather nominals i

$$bound_{\Theta,i}(p) \leq \nu(i,p) \leq bound^{\Theta,i}(p).$$

Then for all subformulas φ of the body of root formula of Θ

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Thus since $bound_{\Theta,i}(p) \leq bound^{\Theta,i}(p)$ by the lower bound less than upper bound lemma, we can use any valuation ν that satisfies

$$bound_{\Theta,i}(p) \leq \nu(i,p) \leq bound^{\Theta,i}(p)$$

for our ν_Θ.

The completeness proof

Theorem (Completeness)

If there is no tableau proof of the formula φ, then there is a model $\mathcal{M} = \langle W, R, \mathbf{n}, \nu \rangle$ and a $w \in W$ such that $\nu(w, \varphi) \neq \top$.

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Proof.

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- Assume there is no tableau proof of φ, and let Θ be an open branch in a saturated tableau starting with $F@_i(\top \rightarrow \varphi)$.

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- Assume there is no tableau proof of φ, and let Θ be an open branch in a saturated tableau starting with $F@_i(\top \rightarrow \varphi)$.
- Construct the model $\mathcal{M}_\Theta = \langle W_\Theta, R_\Theta, \mathbf{n}_\Theta, \nu_\Theta \rangle$.

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- Since $F@_i(\top \rightarrow \varphi) \in \Theta$ it follows by the ordering lemma that $\top \nleq bound^{\Theta,i}(\varphi)$.

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- Construct the model $\mathcal{M}_\Theta = \langle W_\Theta, R_\Theta, \mathbf{n}_\Theta, \nu_\Theta \rangle$.
- Since $F@_i(\top \to \varphi) \in \Theta$ it follows by the ordering lemma that $\top \not\leq bound^{\Theta,i}(\varphi)$.
- But by the main completeness theorem, (since φ is a subformula of the body of the root formula and i is an urfather) it follows that $\nu_\Theta(i, \varphi) \leq bound^{\Theta,i}(\varphi)$.

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- Assume there is no tableau proof of φ, and let Θ be an open branch in a saturated tableau starting with $F@_i(\top \rightarrow \varphi)$.
- Construct the model $\mathcal{M}_\Theta = \langle W_\Theta, R_\Theta, \mathbf{n}_\Theta, \nu_\Theta \rangle$.
- Since $F@_i(\top \rightarrow \varphi) \in \Theta$ it follows by the ordering lemma that $\top \not\leq bound^{\Theta,i}(\varphi)$.
- But by the main completeness theorem, (since φ is a subformula of the body of the root formula and i is an urfather) it follows that $\nu_\Theta(i, \varphi) \leq bound^{\Theta,i}(\varphi)$.
- Thus it follows that $\top \not\leq \nu_\Theta(i, \varphi)$, thus $\nu_\Theta(i, \varphi) \neq \top$. □

Thank you!