Duality theory as a Rosetta Stone

or

Using duality theory to export methods from modal logic

Mai Gehrke

Radboud Universiteit Nijmegen

Overview

0. Introduction

1. Reiterman Theorem and classes of regular languages

2. Relational semantics for substructural logics

3. Extended Stone duality and canonical extension

Modal Logic

Syntax		*Semantics*
Specification language to talk about transition systems	→ ←	internal behavior of transition systems

The logical completeness between modal logic (ML) and state space semantics is at the base of the success of ML

Contributing factors

for the success of ML

- Various decidability results
- A modular theory for a hierarchy of useful languages
- A highly advanced and extensive theory including constructions, theorems, and concepts

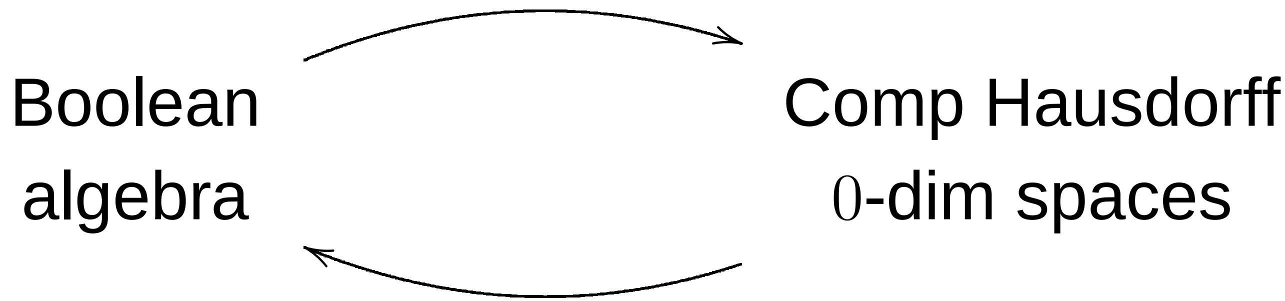
Thesis

(1) The completeness between syntax and semantics in ML is a special case of EXTENDED Stone duality

(2) With a good understanding of this extended duality one may see that the extensive and powerful model theory of ML is actually available VERY WIDELY

Stone duality

Representation theory for Boolean algebra



$$A \hookrightarrow (\mathcal{P}(X), \cap, \cup, (\ \ )^c, \emptyset, X)$$

At the heart of the logical completeness relation between syntactic specification and state space semantics

Additional operations/properties

$\rightarrow$ in IPC $\leftrightsquigarrow$ partial order $\leq$ with properties

$\Box$ in ML $\leftrightsquigarrow$ Kripke relation R

$\Box(\phi) \rightarrow \Box(\Box(\phi))$ $\leftrightsquigarrow$ R is transitive

How does this fit in the general duality picture?

EXTENDED Stone duality

We start with two applications of this thesis before looking into the technical content of our Rosetta Stone

Overview

0. Introduction

1. Reiterman Theorem and classes of regular languages
   [G, Grigorieff & Pin; ICALP'08]

2. Relational semantics for substructural logics

3. Extended Stone duality and canonical extension

Reiterman Theorem

A class $\mathcal{W}$ of finite algebras of a variety $\mathcal{V}$ is closed under

- homomorphic images,

- subalgebras,

- FINITE products,

if and only if $\mathcal{W}$ may be specified by a set of equations $u \approx v$ where u and v are pseudoterms over $\mathcal{V}$.

(The set of pseudoterms is $\widehat{F_{\mathcal{V}}(A)}$, the profinite completion of the free $\mathcal{V}$ algebra over A)

Classes of finite state automata

finite state automata $\rightsquigarrow$ (Kleene) regular language

$\leftrightsquigarrow$ (Myhill/Scott & Rabin) finite monoid

Classes of interest:

- languages with particular properties;
- automata with particular properties

Eilenberg characterized those classes that correspond to pseudovarieties of finite monoids

Star-free languages

Mc Naughton

Star-free languages ⟷ first-order logic

Schützenberger

Aperiodic monoids ⟷ $x^{\omega} \approx x^{\omega+1}$

decidable!

Piecewise testable languages

Thomas

Piecewise testable languages ⟷ BΣ_1 formulas

I. Simon

$\mathcal{J}$-trivial monoids ⟷ $(xy)^\omega \approx (yx)^{\omega+1}$

decidable!

Encompassing more general classes

Several generalizations of Eilenberg's and Reiterman's theorems have been obtained:

- Pin (1995) + Pin-Weil (1996)
- Pippenger (1997)
- Polák (2001)
- Esik (2002), Straubing (2002) + Kunc (2003)

No one of these results provides a common and most general framework for these kinds of results

NOT a modular collection of results

Profinite completions

Theorem: [GGP] Let A be any algebraic structure. Then the profinite completion

$$\widehat{A} \quad \text{WITH} \quad \text{its operations}$$

is the extended dual space of

$$(\mathrm{Rec}(\mathrm{A}), \cap, \cup, (\ \ )^{\mathrm{c}}, \emptyset, \mathrm{A}) \; + \; \text{residuals of complexified ops}$$

$$\mathrm{Rec}(\mathrm{A}) \; = \; \{\phi^{-1}(\mathrm{P}) \mid \phi : \mathrm{A} \to \mathrm{F} \text{ hom, } \mathrm{F} \text{ finite, } \mathrm{P} \subseteq \mathrm{F}\}$$

$$K \cdot L \; \subseteq \; M \quad \Longleftrightarrow \quad L \; \subseteq \; K \backslash M$$

$$\Longleftrightarrow \quad K \; \subseteq \; M/L$$

The dual of $\mathrm{Rec}(A^*)$

The map
$$\begin{aligned} A^* &\hookrightarrow X_A \\ u &\mapsto p_u = \{L \,|\, u \in L\} \end{aligned}$$

embeds $(A^*, \cdot, 1)$ in $(X_A, \cdot, p_1)$ as a discrete submonoid

The identity
$$(H\backslash L)/K = H\backslash(L/K)$$

holds in $\mathrm{Rec}(A^*)$ and dually it corresponds to the associativity of $\cdot$

Theorem: [GGP] The dual space $(X_A, \tau, \cdot)$ of the residuated Boolean algebra $(\mathrm{Rec}(A^*), \cdot, /, \backslash)$ is a totally disconnected compact topological monoid extending $(A^*, \cdot)$

Reitermann's pseudoterms

Duality yields a 1-1 correspondence between continuous monoid morphisms

$$\tilde{\phi} : X_A \to F, \quad F \text{ a finite monoid}$$

and maps

$$\phi : A \to F, \quad F \text{ a finite monoid.}$$

Theorem: [GGP]
The dual space $(X_A, \tau, \cdot)$ of the residuated Boolean algebra $(\mathrm{Rec(A^*)}, \cdot, /, \backslash)$ is Reitermann's space of pseudoterms over A for the algebraic theory of finite monoids

Semigroup invariants for automata

Let $\mathcal{A}$ be a finite state automaton, $L(\mathcal{A})$ the corresponding language. We get

$$RB(\mathcal{A}) = (< L(\mathcal{A}) >, \{a\backslash\ , /a\}_{a \in A})$$

the residuation-closed Boolean subalgebra of $\mathrm{Rec}(\mathrm{A}^*)$ generated by $L(\mathcal{A})$

Theorem: [GGP]
The syntactic monoid of $L(\mathcal{A})$ is the dual space of the algebra $RB(\mathcal{A})$

Classes of languages

$\mathcal{C}$ a class of regular languages closed under $\cap$ and $\cup$

$$\mathcal{C} \quad \hookrightarrow \quad Rec(A^*) \quad \hookrightarrow \quad \mathcal{P}(A^*)$$

DUALLY

$$X_{\mathcal{C}} \quad \twoheadleftarrow \quad \widehat{A^*} \quad \twoheadleftarrow \quad \beta(A^*)$$

That is, $\mathcal{C}$ is described dually EQUATING elements of $\widehat{A^*}$.

This is Reiterman's theorem.

Eilenberg-Reitermann theory

Using the fact that sublattices of $\mathrm{Rec}(A^*)$ correspond to topological quotients of X_A we get a vast generalization of Eilenberg-Reitermann theory
[GGP]

Closed under	Equations	Definition
$\cup, \cap$	$u \to v$	$\hat{\eta}(u) \in \hat{\eta}(L) \Rightarrow \hat{\eta}(v) \in \hat{\eta}(L)$
quotienting	$u \leq v$	for all x, y, $xuy \to xvy$
complement	$u \leftrightarrow v$	$u \to v$ and $v \to u$
quotienting and complement	$u = v$	for all x, y, $xuy \leftrightarrow xvy$
Closed under inverses of morphisms		**Interpretation of variables**
all morphisms		words
non-erasing morphisms		nonempty words
length multiplying morphisms		words of equal length
length preserving morphisms		letters

Overview

0. Introduction

1. Reiterman Theorem and classes of regular languages

2. Relational semantics for substructural logics [Dunn,G & Palmigiano; JSL'05]

3. Extended Stone duality and canonical extension

Sequent calculus for IPC

The logical rules:

$$\frac{}{\phi \vdash \phi}\ \text{(Id)} \qquad \frac{\Gamma \vdash \phi \quad \Delta; \phi; \Sigma \vdash \psi}{\Delta; \Gamma; \Sigma \vdash \psi}\ \text{(Cut)}$$

$$\frac{\phi; \Gamma \vdash \psi}{\Gamma \vdash \phi \to \psi}\ (\to\text{R}) \qquad \frac{\Gamma \vdash \phi \quad \Sigma, \psi \vdash \theta}{\Gamma; \Sigma; \phi \to \psi \vdash \theta}\ (\to\text{L})$$

$$\frac{\Gamma \vdash \phi \quad \Sigma \vdash \psi}{\Gamma; \Sigma \vdash \phi \wedge \psi}\ (\wedge\text{R}) \qquad \frac{\Gamma; \phi \vdash \theta}{\Gamma; \phi \wedge \psi \vdash \theta} \quad \frac{\Gamma; \psi \vdash \theta}{\Gamma; \phi \wedge \psi \vdash \theta}\ (\wedge\text{L})$$

$$\frac{\Gamma \vdash \phi}{\Gamma \vdash \phi \vee \psi} \quad \frac{\Gamma \vdash \psi}{\Gamma \vdash \phi \vee \psi}\ (\vee\text{R}) \qquad \frac{\Gamma; \phi \vdash \theta \quad \Sigma; \psi \vdash \theta}{\Gamma; \Sigma; \phi \vee \psi \vdash \theta}\ (\vee\text{L})$$

Sequent calculus

The structural rules:

$$\frac{\Gamma \; ; \; \Delta \; \vdash \; \psi}{\Gamma \; ; \; \phi \; ; \; \Delta \; \vdash \; \psi}, \qquad \text{(Thinning)}$$

$$\frac{\Gamma \; ; \; \phi \; ; \; \phi \; ; \; \Delta \; \vdash \; \psi}{\Gamma \; ; \; \phi \; ; \; \Delta \; \vdash \; \psi}, \qquad \text{(Contraction)}$$

$$\frac{\Gamma \; ; \; \phi \; ; \; \psi \; ; \; \Delta \; \vdash \; \theta}{\Gamma \; ; \; \psi \; ; \; \phi \; ; \; \Delta \; \vdash \; \theta}, \qquad \text{(Permutation)}$$

Substructural logic

The structural rules as logical rules:

$$\frac{\Gamma \,;\, \Delta \;\vdash\; \psi}{\Gamma \,;\, \phi \,;\, \Delta \;\vdash\; \psi} \qquad \text{(Thinning)}$$

destroys relevance

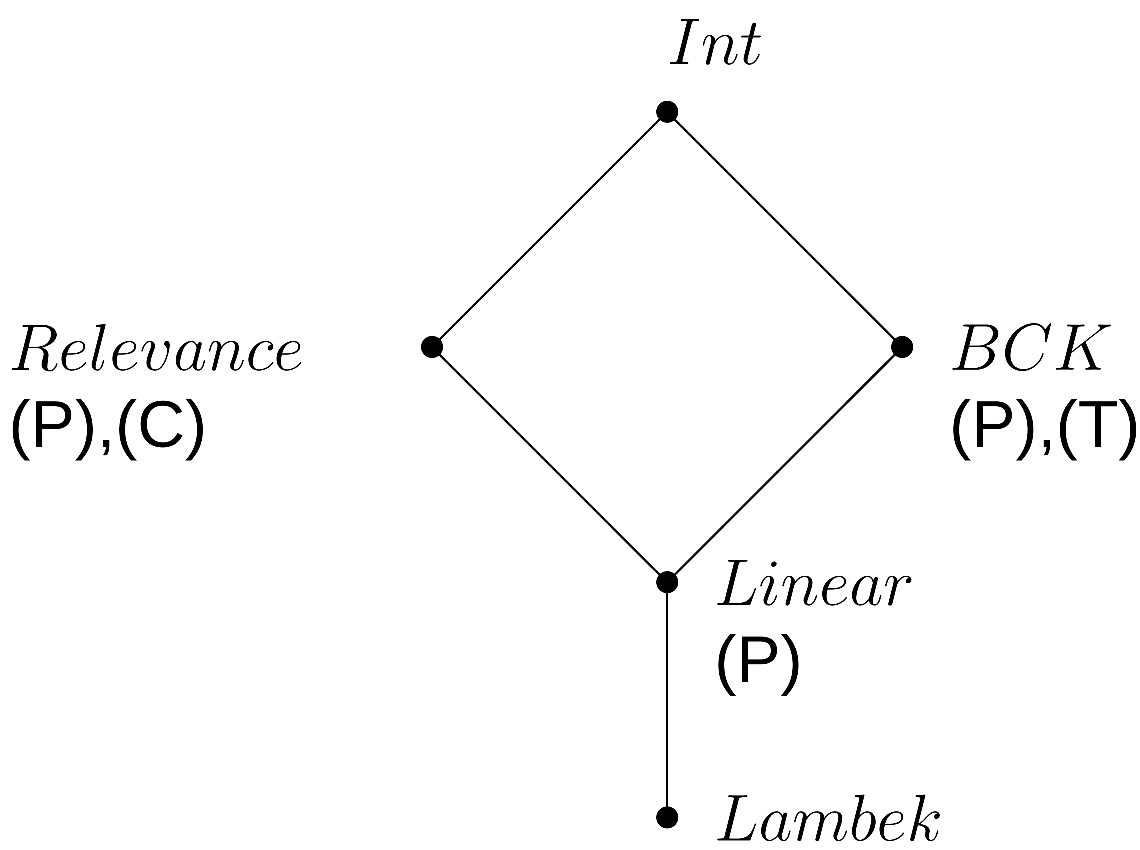
$$\frac{\Gamma \,;\, \phi \,;\, \phi \,;\, \Delta \;\vdash\; \psi}{\Gamma \,;\, \phi \,;\, \Delta \;\vdash\; \psi} \qquad \text{(Contraction)}$$

destroys resource sensitivity

$$\frac{\Gamma \,;\, \phi \,;\, \psi \,;\, \Delta \;\vdash\; \theta}{\Gamma \,;\, \psi \,;\, \phi \,;\, \Delta \;\vdash\; \theta}, \qquad \text{(Permutation)}$$

destroys sensivity to order of execution

The hierarchy of substructural logics



Algebraic semantics

$(A, \leq, \circ, \rightarrow, \leftarrow)$ residuated ordered semigroups

- $$(a \circ b) \circ c \;=\; a \circ (b \circ c)$$
- $$a \circ b \leq c \quad\Longleftrightarrow\quad b \leq a \rightarrow c$$

$$\Longleftrightarrow \quad a \leq c \leftarrow b$$

NB! Lattice models are not necessarily distributive

Additional axioms:

(T) $\quad a \circ b \leq b$

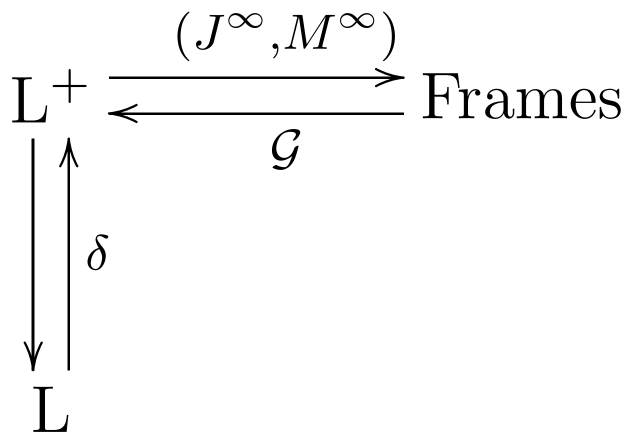
(C) $\quad a \leq a \circ a$

(P) $\quad a \circ b \leq b \circ a$

Uniform and modular relational semantics?

Semantics for substructural logic

[DGP]



Objects in the dual category are $F = (X, Y, \leq)$ with:

$$\forall x\, \forall x'\, (x \neq x' \implies \uparrow x \neq \uparrow x')$$

$$\forall x\, \exists y\, (x \not\leq y \text{ and } \forall x'\, [\uparrow x' \supset \uparrow x \implies x' \leq y])$$

and the dual statements.

Relational semantics

[DGP]

$$F = (X, Y, \leq; R)$$

where $(X, Y, \leq)$ is a frame

$R \subseteq X \times Y \times X$
satisfying

(i) $R[\_, y, x]^{ul}$

(i)' $R[x_1, \_, x_2]^{lu}$

(i)" $R[x, y, \_]^{ul}$

(ii) First order condition for associativity

Uniform and modular

[DGP]

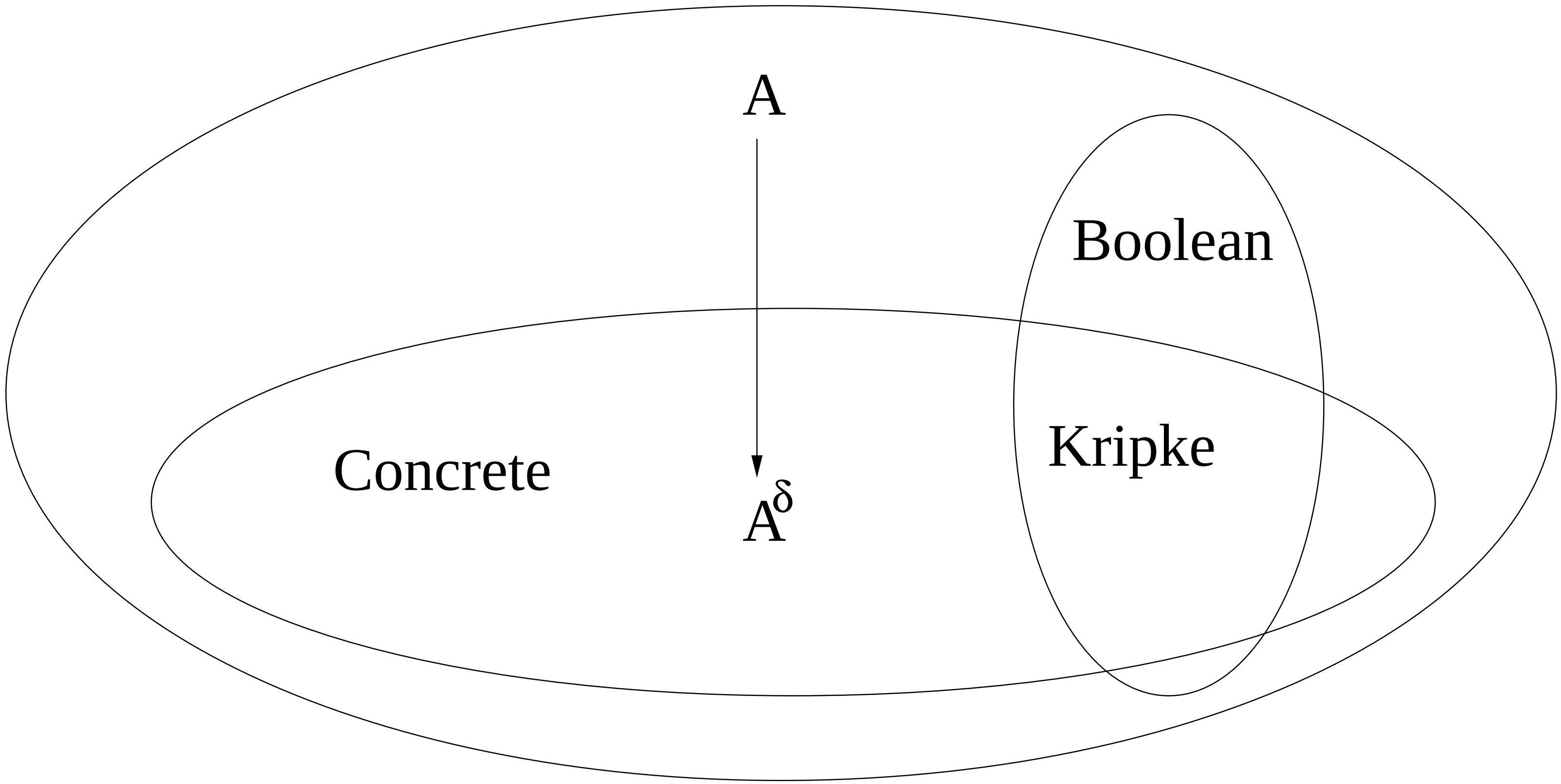
Semantics for substructural logics

(P) if and only if $\forall x_1, x_2, y\,[\,R(x_1, y, x_2) \iff R(x_2, y, x_1)\,]$

(T) if and only if $\forall x_1, x_2, y\,[\,x_1 \leq y \Rightarrow R(x_1, y, x_2)\,]$

(C) if and only if $\forall x, y\,[\,R(x, y, x) \Rightarrow x \leq y\,]$

Classes of algebraic models



Concrete $\leftrightsquigarrow$ F^+ for some $F = (X, Y, \leq, R)$

Kripke $\leftrightsquigarrow$ F^+ for some $F = (X, R)$

Overview

0. Introduction

1. Reiterman Theorem and classes of regular languages

2. Relational semantics for substructural logics

3. Extended Stone duality and canonical extension
   (with Jónsson, Harding, Venema, Priestley, . . .)

Extended Stone duality

[Goldblatt '89]

Boolean algebra	Comp Hausdorff 0-dim spaces
$\Diamond$ n-ary normal op	$R_\Diamond$ $(n+1)$-ary relation point closed pre-im of closed is closed
H,S (P$_{fin}$ $\leq$) P$_{Boo}$, P$_\mu$	genSub, p-morphic image (fin.) Boo. disj. union, ?

Algebraic approach

[Jónsson & Tarski '51]

$$\begin{array}{ccc} A & \stackrel{\alpha}{\hookrightarrow} & (\mathcal{P}(X), \cap, \cup, (\ \ )^c, \emptyset, X) = A^\delta \\ a & \mapsto & \widehat{a} = \{\mu \mid a \in \mu\} \end{array}$$

This completion is characterized by

- α is dense: Every element of A^δ is a join of meets of elements from A (and dually)
- α is compact: For every filtering subset $F \subseteq A$ and any directed subset $I \subseteq A$, if

  $$\bigwedge F \leq \bigvee I \quad \text{in } A^\delta$$

  then $a \leq b$ for some $a \in F$ and $b \in I$

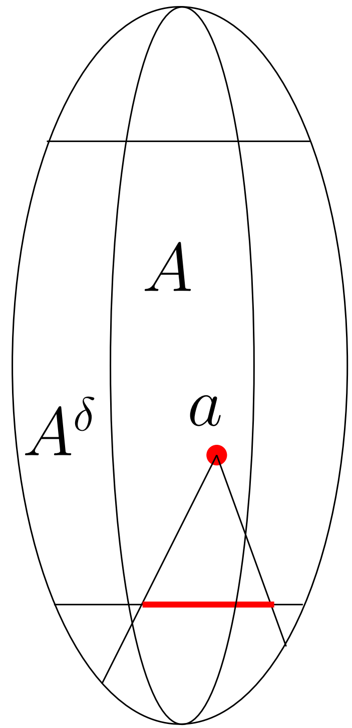
Canonical extensions approach

Both X_A

and A

sit inside

A^δ



$M^\infty(A^\delta) \cong X_A$

$J^\infty(A^\delta) \cong X_A$

$\hat{a} = \{x \mid x \leq a\}$

$J^\infty(A^\delta) \ \subseteq \ K(A^\delta) \ = \$ meet closure of A in A^δ

$M^\infty(A^\delta) \ \subseteq \ O(A^\delta) \ = \$ join closure of A in A^δ

Canonical extension of maps

[Jónsson-Tarski'51; G-Jónsson'04]

$$f^\sigma(u) = \liminf f(u)$$
$$= \bigvee\{\bigwedge f([x,y] \cap L) \mid K(A^\delta) \ni x \leq u \leq y \in O(A^\delta)\}$$

Lower semi-continuous envelope which is $(\delta, \iota^\uparrow)$-continuous

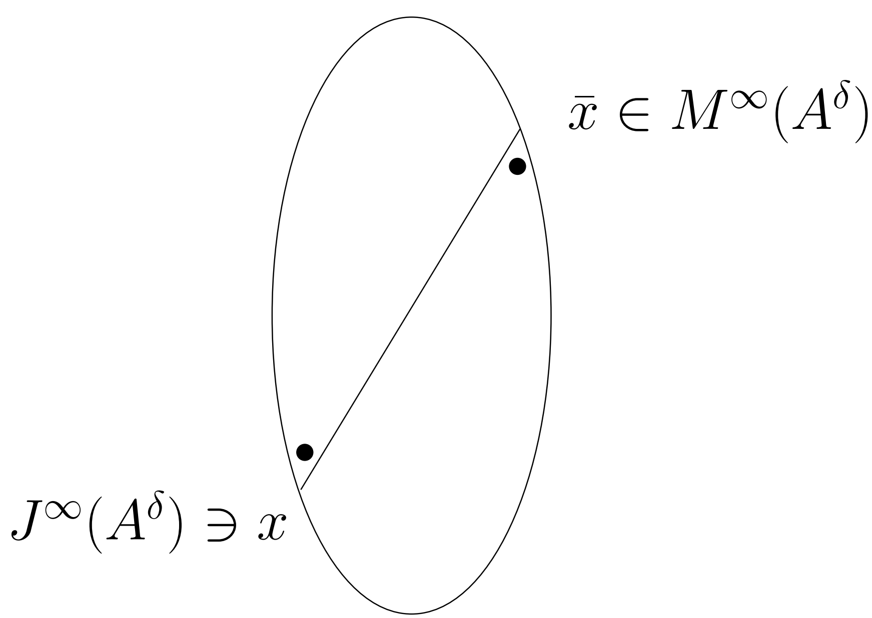
$$f^\pi(u) = \limsup f(u)$$
$$= \bigwedge\{\bigvee f([x,y] \cap L) \mid K(A^\delta) \ni x \leq u \leq y \in O(A^\delta)\}$$

Upper semi-continuous envelope which is $(\delta, \iota^\downarrow)$-continuous

Duals of operations

$$f \quad \mapsto \quad f^\sigma \quad \mapsto \quad f^\sigma \restriction J^\infty(A^\delta)$$
$$= \{(x,y) \mid x \leq f^\sigma(y)\} \;:=\; R_f$$

$$f \quad \mapsto \quad f^\pi \quad \mapsto \quad f^\pi \restriction M^\infty(A^\delta)$$
$$= \{(x,y) \mid \overline{x} \leq f^\pi(\overline{y})\} \;:=\; S_f$$



Applications within ML

- Algebraic version of the Fine-van Benthem-Goldblatt theorem:

  If $\mathcal{K}$ is a class of MAs so that $\mathsf{P}_\mu(\mathcal{K})$ is canonical, then $\mathcal{V}(\mathcal{K})$ is canonical [G & Jónsson '99 or '04]

- Algebraic version of the crucial lemma in this connection:

  Given any MA A, there is a non-standard extension A^*, so that

  $$A^\delta \hookrightarrow DM(A^*)$$

  [G,Harding, Venema; TAMS '05]

  Corollary: If a variety is closed under MacNeille completion, then it is closed under canonical extension

OBS

This theory just depends on the existence of a dense and compact completion and some infinite distributivity

Distributivity in canonical extensions:

[G & Harding; JAlg ’01]

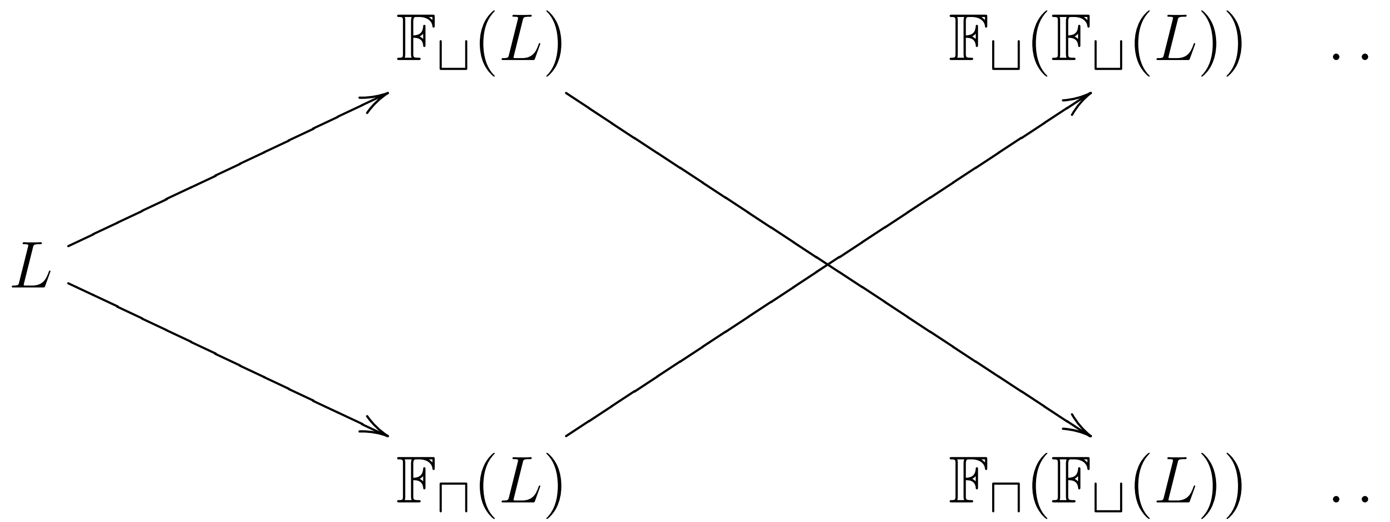
If F_i is filtering in A for each $i \in I$ then in A^δ

$$\bigvee_i (\bigwedge F_i) = \bigwedge_{\Phi : I \to \bigcup F_i} \bigvee Im(\Phi)$$

choice funct.

and dually

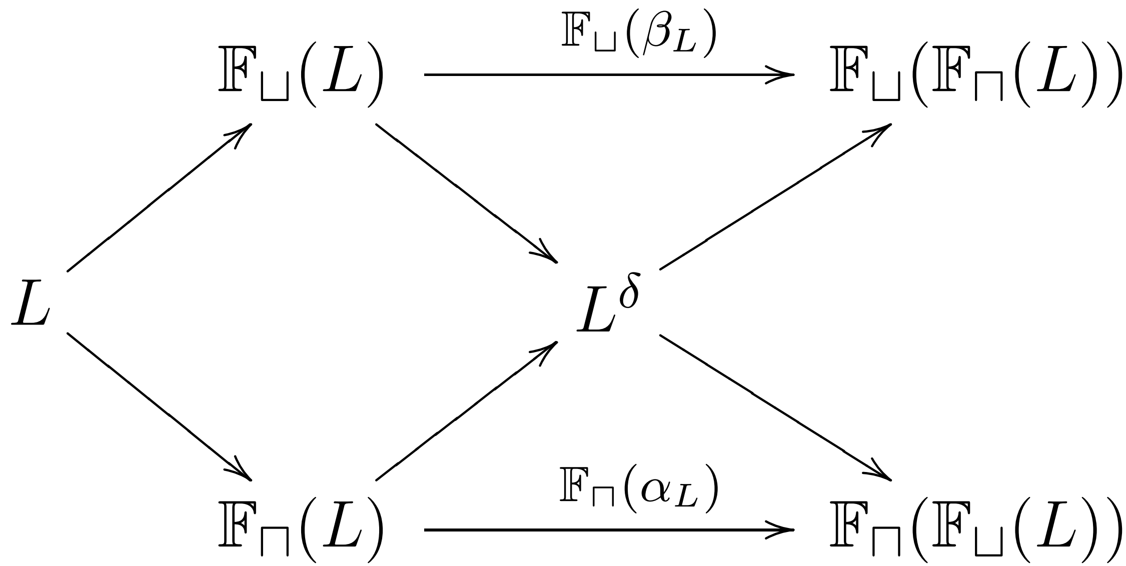
Hierarchy of completions



- In general there is no free 2-sided completion
- What is the 2-sided interpolant, the Δ_i-completion, for $i = 1, 2, \ldots$?

Δ_1-completion

[G & Priestley; MLQ ’08]



Δ_0-completion $=$ Dedekind-MacNeille completion

Δ_1-completion $=$ Jónsson-Tarski canonical completion

Existence of canonical extension

- For BAOs [Jónsson & Tarski ’51]
- For Stone algebras [Comer ’89]
- For DLOs [G & Jónsson ’94]
- For DLEs [G & Jónsson ’99]
- For LEs [G & Harding ’01]
- For POMs [Dunn, G & Palmigiano ’05]