Undecidability for Arbitrary Public Announcement Logic

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Outline

Arbitrary Public Announcement Logic

Arbitrary Public Announcement Logic (*APAL*) is a dynamic logic that extends epistemic logic with

- a *public announcement* operator to represent the update corresponding to a public announcement;
- an *arbitrary announcement* operator that quantifies over annoucements.

APAL was introduced by Balbiani, Baltag, van Ditmarsch, Herzig, Hoshi and de Lima in 2007 (TARK) as an extension of public announcement logic. An journal version ('Knowable' as 'known after an announcement') is forthcoming in the *Review of Symbolic Logic*.

Syntax of APAL

The formulas of *APAL*, $\mathcal{L}_{apal}$, are inductively defined as

$$\varphi ::= \ p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid K_a\varphi \mid [\varphi]\varphi \mid \Box\varphi$$

where a is taken from the set of agents A;
and where p is taken from the set of atomic propositions P.

$[\varphi]\psi$ means "*after public announcement of* φ, ψ *is true*".

$\Box\psi$ means "*after public announcement of any* φ, ψ *is true*".

Semantics of APAL

The structures are pointed Kripke models $M = (S, \sim, V)$ with equivalence relations $\sim_a$ for each agent.

$$
\begin{array}{lll}
M, s \models p & \text{iff} & s \in V(p) \\
M, s \models \neg\varphi & \text{iff} & M, s \not\models \varphi \\
M, s \models \varphi \wedge \psi & \text{iff} & M, s \models \varphi \text{ and } M, s \models \psi \\
M, s \models K_a\varphi & \text{iff} & \text{for all } t \in S : s \sim_a t \text{ implies } M, t \models \varphi \\
M, s \models [\varphi]\psi & \text{iff} & M, s \models \varphi \text{ implies } M|\varphi, s \models \psi \\
M, s \models \Box\varphi & \text{iff} & \text{for all epistemic } \psi : M, s \models [\psi]\varphi
\end{array}
$$

$M|\varphi$: restriction of M to the states where φ is true.

$M|\varphi = (S', \sim', V')$ such that:
$S' = \{s \in S \mid M, s \models \psi\}$;
for all $a \in A$, $\sim'_a = \sim_a \cap \ (S' \times S')$;
and for all $p \in P$, $V'(p) = V(p) \cap S'$.

Example: $\Diamond(K_a p \vee K_a \neg p)$ is valid

$\Diamond\varphi$ is true in a model, iff
there is an epistemic ψ such that $\langle\psi\rangle\varphi$ is true, iff
there is a model restriction such that φ is true in the restriction.

$\underline{1}$——0 $\Rightarrow_{p}$ $\underline{1}$

$\Diamond(K_a p \vee K_a \neg p), \langle p\rangle(K_a p \vee K_a \neg p)$ $p, K_a p$

1——$\underline{0}$ $\Rightarrow_{\neg p}$ $\underline{0}$

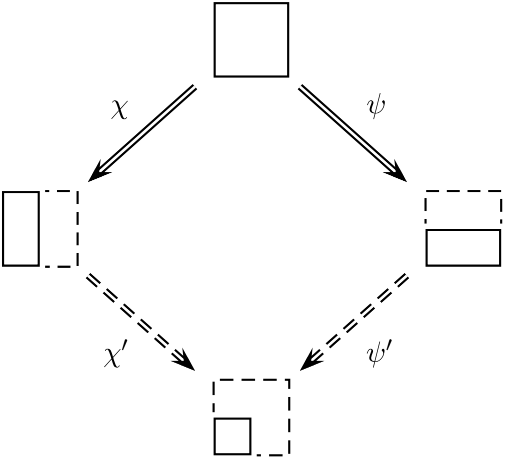
$\Diamond(K_a p \vee K_a \neg p), \langle \neg p\rangle(K_a p \vee K_a \neg p)$ $\neg p, K_a \neg p$

Validities: transitivity / 4 $\models \Box\varphi \rightarrow \Box\Box\varphi$

A sequence of two announcements is again an announcement.

$$[\psi][\chi]\varphi \leftrightarrow [\psi \wedge [\psi]\chi]\varphi$$

Validities: Church-Rosser $\models \Diamond\Box\varphi \rightarrow \Box\Diamond\varphi$



Various results for APAL

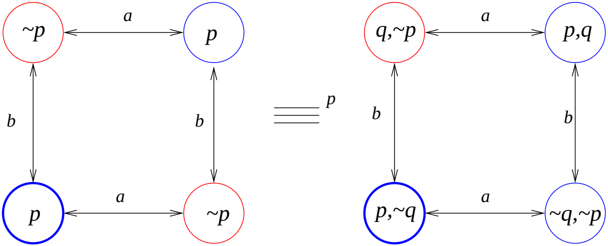
- sound and complete axiomatization [Balbiani et al. 2007/8]
- sound tableau method [Balbiani et al. 2007b TABLEAUX]
- bisimulation invariant
- non-compact
- more expressive than epistemic logic
- complexity of model checking is PSPACE-complete [Agotnes et al. 2008 manuscript]

Here we address the question of decidability.

Q-Bisimulation Invariance

Although *APAL* is bisimulation invariant it does **not** satisfy a related property: if two worlds are bisimilar up to some set of atoms Q, they agree on the interpretation of formulas that contain only the atoms Q.

Counterexample: The pointed models below are p-bisimilar, but formula $\Diamond(K_b p \wedge \neg K_a K_b p)$ is false on the left and true on the right.



Tilings

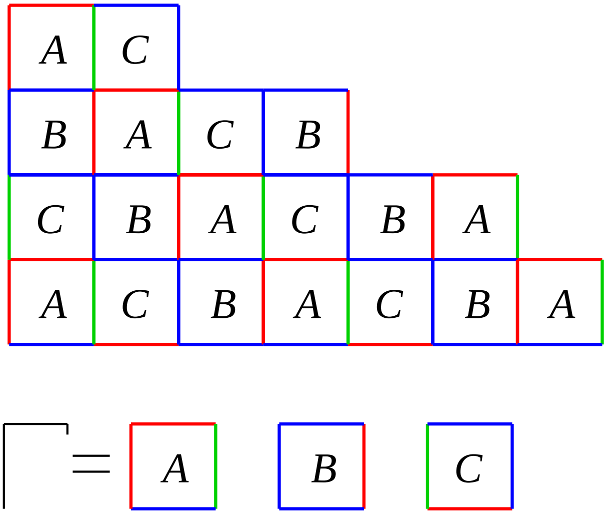
Definition

Let C be a finite set of *colours* and define a C-tile to be a four-tuple over C, $\gamma = (\gamma^t, \gamma^r, \gamma^f, \gamma^\ell)$, where the elements are referred to as, respectively, *top, right, floor* and *left*. The tiling problem is, for any given finite set of C-tiles, Γ, to determine if there is a function $\lambda : \omega \times \omega \longrightarrow \Gamma$ such that for all $(i,j) \in \omega \times \omega$:

1. $\lambda(i,j)^t = \lambda(i,j+1)^f$
2. $\lambda(i,j)^r = \lambda(i+1,j)^\ell$.

The tiling problem has been shown to be co-RE complete by Harel [J ACM 1986].

A Tiling



Tilings and Modal Logic

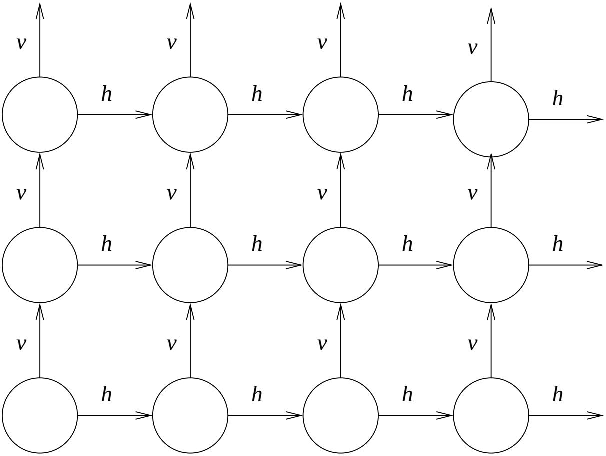
- Tilings are frequently used to demonstrate complexity results for modal logics.
- The possible worlds of modal logic correspond to the points on the grid.
- The accessibility relation corresponds to the adjacency relation of the grid.
- The propositions are used to indicate which tile labels which point on the grid.
- To show *APAL* is undecidable we show that for any given set of tiles, Γ, we can define a formula $Tile_\Gamma$ that is satisfiable if and only if Γ can tile the plane.

Some Simplifying Assumptions

- The paper uses five agents (one for each direction and one for the transitive closure), two atoms, for marking the grid like a chess board, and an atom for each tile in the set Γ.
- For simplicity we shall use only three agents h, v and t (called *horizontal*, *vertical* and *transitive*), where the relations for h and v are serial, and that the relation for t is the reflexive transitive closure of the relations for h and v.
- The encoding will use a distinct proposition for each tile as well as the labels *black* (B) and *white* (W).
- *The additional complexity in the paper is related to the fact epistemic relations are equivalence relations. This doesn't alter the proof. It merely complicates it*

Grid-like model

A model is grid-like if the relations for *h* and *v* correspond to the vertical and horizontal adjacency relations of the plane.



Exploiting the absence of distinguishing propositions.

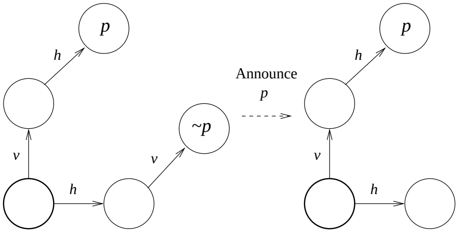
- In order to constrain the model so that it is grid-like, we exploit the *arbitrary announcement operator*, and particularly the lack of distinguishing propositions.
- For example, a crucial property of a grid is that, if $u \longrightarrow_v w \longrightarrow_h v$, and $u \longrightarrow_h w' \longrightarrow_v v'$, then $v = v'$.
- We can express a slightly weaker property: *if* $u \longrightarrow_v w \longrightarrow_h v$*, then for some* v'*,* $u \longrightarrow_h w' \longrightarrow_v v'$*, and* v *and* v' *agree on the interpretation of every modal formula*.

Distinguishing Propositions

We can express a weaker property: $M, u \models C1 \wedge C2$

$$C1 = \Box(L_v L_h \top \rightarrow K_h L_v \top)$$
$$C2 = \Box(L_h L_v \top \rightarrow K_v L_h \top)$$

These formulas show that *every* announcement that is true at a $\rightarrow_v \rightarrow_h$ reachable world is true at some $\rightarrow_h \rightarrow_v$-reachable world.



n-Q-Bisimilarity

Definition

Let $n \in \omega$ and let Q be a finite set of atomic propositions. We say two worlds u and v are *n-Q-bisimilar* if they agree on every formula of modal logic using only the atoms in Q, and only having modal operators nested up to depth n.

Lemma

Let Q be fixed. For every world u, there is some modal formula, φ_u^n such that for all v, $M, v \models \varphi_u^n$ if and only if $u \simeq_n^Q v$.

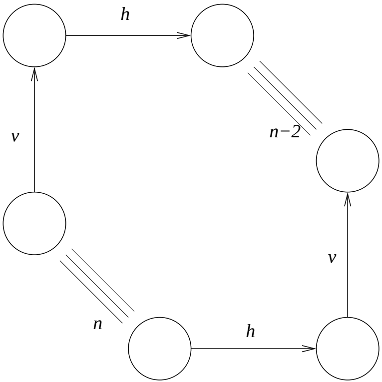
Lemma

If two worlds agree on every modal formula, then they are n-Q-bisimilar for every n and every Q.

Key Lemma

Lemma

Suppose $u \cong^Q_n u'$, where $n \geq 2$, $M, u \models C1 \wedge C2$, and $u \longrightarrow_v w \longrightarrow_h v$, and $u' \longrightarrow_h w' \longrightarrow_v v'$. Then $v \cong^Q_{n-2} v'$.



Encoding the Tiling

The existence of a tiling can now be encoded as follows:

$$
\begin{array}{rcl}
\mathit{cheq} & = & (B \to (K_v W \wedge K_h W)) \wedge (W \to (K_v B \wedge K_h B)) \\
C1 & = & \Box(L_v L_h \top \to K_h L_v \top) \\
C2 & = & \Box(L_h L_v \top \to K_v L_h \top) \\
\mathit{tile} & = & \left[\begin{array}{l} \bigvee_{\gamma \in \Gamma} \gamma \wedge \\ \bigwedge_{\gamma \in \Gamma} \left(\gamma \to K_v \bigvee_{\gamma^r = \delta^\ell} \delta \right) \wedge \\ \bigwedge_{\gamma \in \Gamma} \left(\gamma \to K_h \bigvee_{\gamma^t = \delta^f} \delta \right) \end{array} \right] \\
\mathit{Tile}_\Gamma & = & B \wedge K_t(\mathit{cheq} \wedge C1 \wedge C2 \wedge \mathit{tile})
\end{array}
$$

Soundness

It is easy to show the construction is sound. Given a tiling-function λ that maps $\omega \times \omega$ to Γ, we can construct a model satisfying $Tile_\Gamma$.

$$
\begin{array}{ccccccc}
\lambda(0,3) & \xrightarrow{h} & \lambda(1,3) & \xrightarrow{h} & \lambda(2,3) & \xrightarrow{h} & \lambda(3,3) \\
v\uparrow & & v\uparrow & & v\uparrow & & v\uparrow \\
\lambda(0,2) & \xrightarrow{h} & \lambda(1,2) & \xrightarrow{h} & \lambda(2,2) & \xrightarrow{h} & \lambda(3,2) \\
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\end{array}
$$

Completeness

- Completeness is harder to show, as we have not managed to enforce that the model is “grid-like”.
- Rather than equality between worlds, we have the weaker notion of n-Q-bisimilarity.
- We are able to use this to show that if $Tile_\Gamma$ is satisfiable, then for every n, we are able to construct a tiling of the $n \times n$ grid.
- Then applying König’s Lemma, we are able to conclude that a tiling of the full plane exists.

Constructing an $n \times n$ grid: Base Case

Suppose that $M, u \models Tile_\Gamma$.

- We define a function $g : \omega \times \omega \to S$
- Rather than requiring $g(i,j) \to_h g(i+1,j)$ we simply require that $g(i,j) \to_h v$ for some v such that $g(i+1,j) \cong_{2n-(i+j)} v$.
- ...and likewise for $\to_v$
- As a base case we select the world u to be $g(0,0)$.
- Clearly u is $2n$-bisimilar to itself.

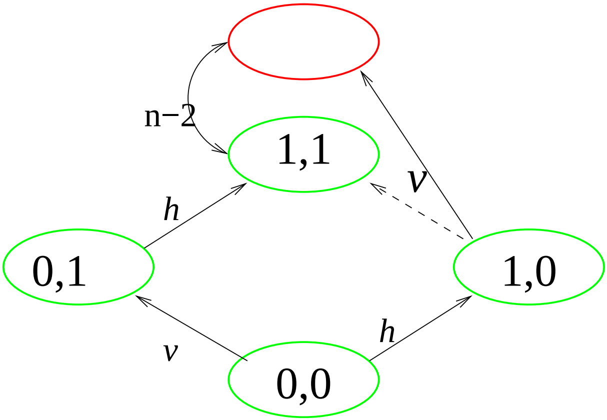
Constructing an $n \times n$ grid: Induction

Suppose that $g(i,j) \cong_k^Q v$, $g(i,j) \rightarrow_v u_1$ and $v \rightarrow_h u_2$.

- by Lemma 5, for *every* w where $u_1 \rightarrow_h w$ there is some $w' \cong_{k-2} w$ where $u_2 \rightarrow_v w'$.
- We can select
  1. $g(i,j+1) = u_1$
  2. $g(i+1,j) = u_2$
  3. $g(i+1,j+1) = w$

  and proceed with the induction.

Constructing an $n \times n$ grid: Induction



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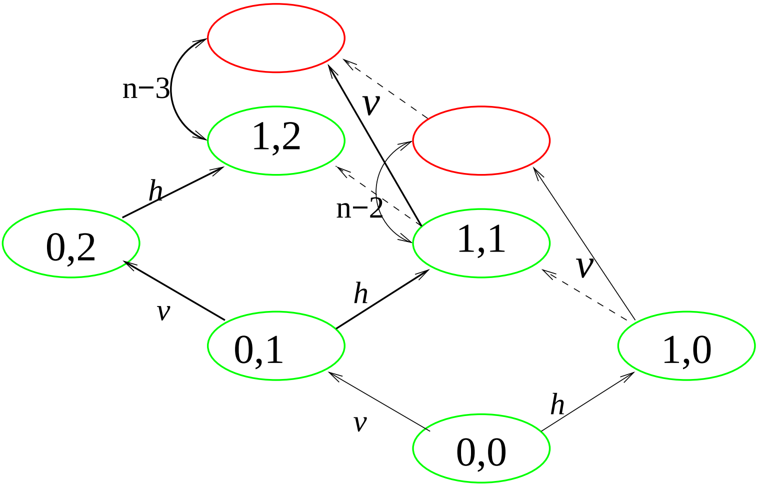

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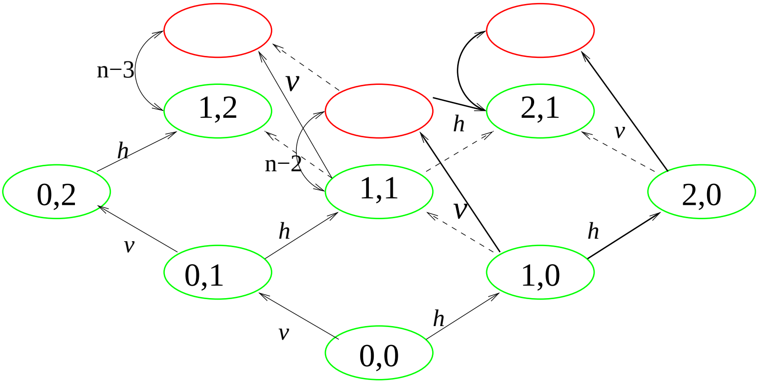

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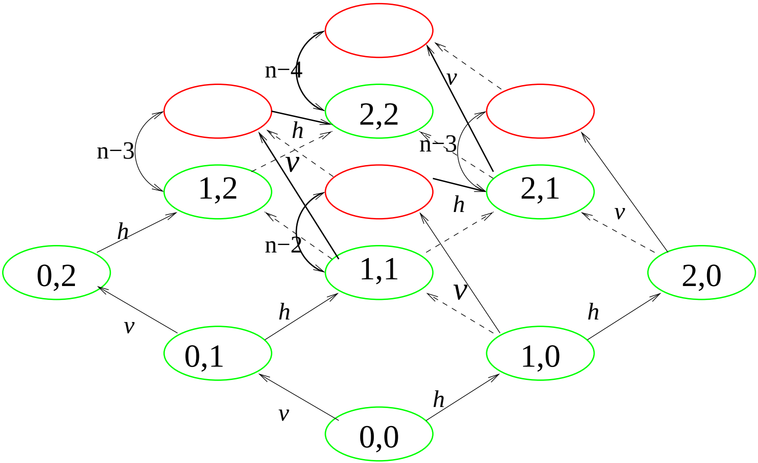


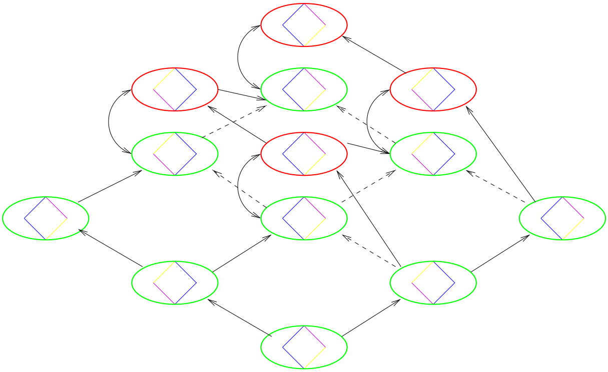
Completing the Tiling

If $M, u \models Tile_\Gamma$.

- for every n we can construct a function $g : [0, n] \times [0, n] \rightarrow S$, where
  1. $g(i,j) \rightarrow_h v$ for some v such that $g(i+1,j) \cong_{2n-(i+j)} v$
  2. $g(i,j) \rightarrow_v v$ for some v such that $g(i,j+1) \cong_{2n-(i+j)} v$
- This is sufficient to show that, for all $i, j < n$, the edges of the tiles match.
- Thus for every n an $n \times n$ tiling exists
- Thus Γ tiles the $\omega \times \omega$ plane.

The Final Colouring

Because any worlds that are at least 0-bisimilar agree on the tile, the adjacent edges of the tiles must match.



Satisfiability is Undecidable

- If $M, u \models Tile_\Gamma$, then we are able to create an $n \times n$ grid for n, and tile it with Γ.
- Thus Γ is able to tile the plane, and hence the satisfiability problem is at least co-RE.
- Since the validity problem for *APAL* is known to be recursively enumerable (via the axiomatization), we have:

Theorem

APAL is co-RE complete.

Syntactic Restrictions

1. Full automated reasoning cannnot be feasible for *APAL*, but can we apply it to interesting restrictions.
2. One such candidate would be to consider restricting the scope of the arbitrary announcement operator, so that it only quantified over announcements of a bounded modal depth, or perhaps was only able to quantify over positive knowledge statements (where every modal operator was in the scope of an even number of negations).
3. A further alternative is to weaken the semantics so that they *are* invariant to *Q*-bisimulations.

Future Event Logic

[vDitmarsch & French, TARK 2008 / KRAMAS 2008]

Language:

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid K_a\varphi \mid G\varphi$$

Semantics:

$$M, s \models G\varphi \quad \text{iff} \quad \text{for all } (M', s') : (M, s) \leftharpoondown\!\!\!\!= (M', s') \text{ implies } M', s' \models \varphi$$

$G\varphi$ is true in an epistemic state iff φ is true in all its *refinements*.

- $\leftrightarrows$ bisimulation: atoms, forth, back —- $\cong$
- $\rightleftharpoons$ simulation: atoms, forth
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Results for *FEL*

1. action model execution is a refinement
2. decidable (and for extensions too)
3. expressivity known
   (via encoding to bisimulation quantified logics, roughly comparable with *mu*-calculus)
4. complexity open
5. axiomatization open and hard
   (in quantifiying over a more general set of announcements we sacrifice the *witnessing formulas* that were used in the APAL axiomatization)

To be continued... Thanks!

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