Generalised Kripke semantics for the Lambek-Grishin calculus

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Outline

Introduction: generalised Kripke semantics

Suggested in (Gehrke, 2006):

Kripke semantics	**generalised Kripke semantics**
domains of application:	
for modal logics	for a broader setting
underlying algebraic structure:	
powerset of a set of worlds	lattice
interpretant of a logical formula:	
arbitrary set of worlds	is described by its worlds and 'information quanta'

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First mentioned in (Grishin, 1983).

LG is a substructural logic extending the implication-fusion fragment (the non-associative Lambek calculus):

- product residuated family of connectives
- plus residuated family of connectives
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A **polarity** is a triple $F = (X, Y, \leqslant)$, where:

- X and Y are non-empty sets (**worlds** and **co-worlds**)
- $\leqslant \subseteq X \times Y$ is a binary relation from X to Y
- $x \leqslant y$: a co-world y is a **part of the world** x

A polarity yields a Galois connection:

$$\begin{array}{rcrcl} (\_)^{\leqslant} & : & \mathcal{P}(X) & \rightarrow & \mathcal{P}(Y) \\ & & A & \mapsto & \{y \in Y \mid \forall x \in X\ (x \in A \Rightarrow x \leqslant y)\} \\ (\_)^{\geqslant} & : & \mathcal{P}(Y) & \rightarrow & \mathcal{P}(X) \\ & & B & \mapsto & \{x \in X \mid \forall y \in Y\ (y \in B \Rightarrow x \leqslant y)\} \end{array}$$

Interpretants are Galois-stable subsets of X:

$$\mathcal{G}(F) = \{A \subseteq X \mid A = (A^{\leqslant})^{\geqslant}\}$$

Define:

- $\uparrow x = \{y \in Y \mid x \leqslant y\}$
- $\downarrow y = \{x \in X \mid x \leqslant y\}$

A polarity F is an **S-frame (a separating frame)** if:

- $\forall x_1, x_2 \in X \; (\uparrow x_1 = \uparrow x_2 \;\Rightarrow\; x_1 = x_2)$
- $\forall y_1, y_2 \in Y \; (\downarrow y_1 = \downarrow y_2 \;\Rightarrow\; y_1 = y_2)$

An S-frame F is an **RS-frame** (a **reduced frame**) if:

- $\forall x \, \exists y \left(x \not\leqslant y \;\wedge\; \forall x' \, (\uparrow x \subset \uparrow x' \Rightarrow x' \leqslant y) \right)$
- $\forall y \, \exists x \left(x \not\leqslant y \;\wedge\; \forall y' \, (\downarrow y \subset \downarrow y' \Rightarrow x \leqslant y') \right)$

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A generalised Kripke model

- a generalisation of a set of worlds: an RS-frame
- a valuation function $V\colon Var \to \mathcal{G}(F)$
  (F is an RS-frame; Var is a set of variables)
- an interpretation is encoded by two relations:
  - a satisfaction relation $\Vdash$
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Non-associative Lambek calculus

Set of formulae:

$$\begin{array}{rclll} \mathcal{F} & = & a, b, c, \ldots \in \mathit{Var} & | & \text{the set of variables} \\ & & A \otimes B,\ A \backslash B,\ B / A & | & \text{product, left/right division} \end{array}$$

Rules ($A, B, C \in \mathcal{F}$):

- an axiom scheme: $A \vdash A$
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Models for non-associative Lambek calculus

To model the behaviour of connectives: add a ternary relation $R_\otimes \subseteq X \times X \times Y$ satisfying compatibility conditions:

$$\begin{array}{llll} \forall x_1, x_2 \in X & (R_\otimes(x_1, x_2, \_)^{\geqslant})^{\leqslant} & = & R_\otimes(x_1, x_2, \_) \\ \forall x_1 \in X, y \in Y & (R_\otimes(x_1, \_, y)^{\leqslant})^{\geqslant} & = & R_\otimes(x_1, \_, y) \\ \forall x_2 \in X, y \in Y & (R_\otimes(\_, x_2, y)^{\leqslant})^{\geqslant} & = & R_\otimes(\_, x_2, y) \end{array}$$

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Models for non-associative Lambek calculus-2

A **model** is a triple $\mathcal{M} = (F, R_{\otimes}, V)$ where:

- $F = (X, Y, \leqslant)$ is an RS-frame
- $R_{\otimes}$ is a compatible relation
- $V : \mathit{Var} \to \mathcal{G}(F)$ is a valuation of variables

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Models for non-associative Lambek calculus-3

Relations $\Vdash \subseteq X \times \mathcal{F}$ and $\succ \subseteq Y \times \mathcal{F}$:

- for $p \in Var$: $x \Vdash p \;\Leftrightarrow\; x \leqslant V(p)$ and $y \succ p \;\Leftrightarrow\; V(p) \leqslant y$
- the relations for complex formulae:
  - $y \succ A \otimes B \;\Leftrightarrow\; \forall x_1, x_2 \left((x_1 \Vdash A \;\wedge\; x_2 \Vdash B) \Rightarrow R_\otimes(x_1, x_2, y)\right)$
  - $x \Vdash A \otimes B \;\Leftrightarrow\; \forall y \,(y \succ A \otimes B \Rightarrow x \leqslant y)$
  - $x \Vdash A \backslash B \;\Leftrightarrow\; \forall x' \,\forall y \left((x' \Vdash A \;\wedge\; y \succ B) \Rightarrow R_\otimes(x', x, y)\right)$
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Soundness and completeness theorem

For $A, B \in \mathcal{F}$ the sequent $A \vdash B$ is derivable iff it is valid over the class of all models.

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LG (definition)

Set of formulae:

$$\begin{array}{rcll} \mathcal{F} & = & a, b, c, \ldots \in Var & \mid \text{the set of variables} \\ & & A \otimes B,\ A \backslash B,\ B / A & \mid \text{product, left/right division} \\ & & A \oplus B,\ A \obslash B,\ B \oslash A & \mid \text{plus, left/right difference} \end{array}$$

Rules ($A, B, C \in \mathcal{F}$):

- Minimal Lambek-Grishin calculus ($LG_\emptyset$):
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Models for $LG_\emptyset$

To model the behaviour of the plus family of connectives: add another ternary relation $R_\oplus \subseteq X \times Y \times Y$:

$$\begin{array}{lllll} \forall y_1, y_2 \in Y & (R_\oplus(\_, y_1, y_2)^{\leqslant})^{\geqslant} & = & R_\oplus(\_, y_1, y_2) \\ \forall y_1 \in Y, x \in X & (R_\oplus(x, y_1, \_)^{\geqslant})^{\leqslant} & = & R_\oplus(x, y_1, \_) \\ \forall y_2 \in Y, x \in X & (R_\oplus(x, \_, y_2)^{\geqslant})^{\leqslant} & = & R_\oplus(x, \_, y_2) \end{array}$$

Models for $LG_\emptyset$-2

The truth conditions for the plus connectives:

- $x \Vdash A \oplus B \;\Leftrightarrow\; \forall y_1, y_2 \left((y_1 \succ A \;\wedge\; y_2 \succ B) \Rightarrow R_\oplus(x, y_1, y_2)\right)$
- $y \succ A \oplus B \;\Leftrightarrow\; \forall x\, (x \Vdash A \oplus B \Rightarrow x \leqslant y)$
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- $x \Vdash A \oslash B \;\Leftrightarrow\; \forall y\, (y \succ A \oslash B \Rightarrow x \leqslant y)$

Soundness and completeness theorem

For $A, B \in \mathcal{F}$ the sequent $A \vdash B$ is derivable in $LG_\emptyset$ iff it is valid over the class of all models.

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Soundness and completeness theorem

For $A, B \in \mathcal{F}$ the sequent $A \vdash B$ is derivable in $LG_\emptyset$ iff it is valid over the class of all models.

Interaction postulates

Interactions involving only product connectives are well-known:

non-associative Lambek calculus + $A \otimes (B \otimes C) \dashv\vdash (A \otimes B) \otimes C$
$\Rightarrow$ associative Lambek calculus

Mixed postulates form 2 groups (Grishin, 1983).

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Interaction postulates-2

The additional postulates come in groups of six that are mutually interderivable in $LG_\emptyset$.

Example:

$$
\begin{array}{llll}
(1) & (B \backslash C) \oslash A \vdash B \backslash (C \oslash A) & (4) & (A \backslash C) \obslash B \vdash C \obslash (A \otimes B) \\
(2) & B \backslash (C \oplus A) \vdash (B \backslash C) \oplus A & (5) & (A \oplus B) / C \vdash A / (C \oslash B) \\
(3) & A \otimes (C \oslash B) \vdash (A \otimes C) \oslash B & (6) & A \oslash (B \obslash C) \vdash (C / A) \backslash B
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Example:

(1)	$(B \backslash C) \oslash A \vdash B \backslash (C \oslash A)$	(4)	$(A \backslash C) \obslash B \vdash C \obslash (A \otimes B)$
(2)	$B \backslash (C \oplus A) \vdash (B \backslash C) \oplus A$	(5)	$(A \oplus B) / C \vdash A / (C \oslash B)$
(3)	$A \otimes (C \oslash B) \vdash (A \otimes C) \oslash B$	(6)	$A \oslash (B \obslash C) \vdash (C / A) \backslash B$

Frame conditions for interaction postulates

Example: postulate number (2) from the family of postulates given above:

$$B \backslash (C \oplus A) \vdash (B \backslash C) \oplus A$$

The condition (in prenex normal form):

$$\forall x \; y_1 \; y_2 \; \exists x_1 \; \forall x_2 \; y_3 \; \exists x_3$$
$$\left(\left[\left(R^{\downarrow}_{\otimes}(x_2, x, x_3) \Rightarrow R_{\oplus}(x_3, y_3, y_1)\right) \Rightarrow R_{\otimes}(x_2, x_1, y_3)\right] \Rightarrow x_1 \leqslant y_2\right)$$
$$\Rightarrow R_{\oplus}(x, y_2, y_1)$$

Here $R^{\downarrow}_{\otimes} \subseteq X \times X \times X$:

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Correspondence-1

To sketch the proof of the correspondence:

- We work in $\mathcal{G}(X, Y, \leqslant)$
- A, B, C are general elements of $\mathcal{G}(X, Y, \leqslant)$
- $\otimes, /, \backslash$ and $\oplus, \oslash, \obslash$ are operations on $\mathcal{G}(X, Y, \leqslant)$ specified by the truth conditions
- $A \vdash B$ means $A \leqslant B$
- $x \Vdash A$ means $x \leqslant A$
- $y \succ A$ means $A \leqslant y$

Correspondence-2

- The interaction axiom: $\forall A, B, C \left(B \backslash (C \oplus A) \leqslant (B \setminus C) \oplus A \right)$
- The axiom holds in $\mathcal{G}(X, Y, \leqslant)$ iff

$$\forall x\ \forall A, B, C \left(x \leqslant B \backslash (C \oplus A) \ \Rightarrow\ x \leqslant (B \setminus C) \oplus A \right)$$

- By residuation:

$$\forall x\ \forall A, B, C \left(B \leqslant (C \oplus A) \,/\, x \ \Rightarrow\ x \leqslant (B \setminus C) \oplus A \right)$$

- Equivalent to $\forall x\ \forall A, C \left(x \leqslant \left([(C \oplus A) \,/\, x] \setminus C \right) \oplus A \right)$

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Correspondence-3

- $\forall x\, \forall A, C\, \Big(x \leqslant \Big([(C \oplus A) \,/\, x] \setminus C\Big) \oplus A\Big)$
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- $\forall x, y_1, y_2\, \forall C\, \Big[[(C \oplus y_1) \,/\, x] \setminus C \leqslant y_2 \;\Rightarrow\; x \leqslant y_2 \oplus y_1\Big]$

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Correspondence-4

- $\ldots \forall x, y_1, y_2\ \forall C \left[\forall x_1 \left(x_1 \leqslant ([C \oplus y_1] / x) \setminus C \;\Rightarrow\; x_1 \leqslant y_2\right) \;\Rightarrow\; x \leqslant y_2 \oplus y_1\right]$
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Conclusions

In this paper:

- generalised Kripke semantics for the minimal calculus $LG_\emptyset$ (soundness and competeness theorem is analogous)
- correspondence result obtained for one interaction postulate

Questions for future research:

- Correspondence results for other interaction axioms
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