Proof theory of the coalgebraic cover modality

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AiML 2008
Nancy

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- this talk - the basic case - the finitary power set functor $\mathcal{P}_\omega$
- coalgebras for $\mathcal{P}_\omega$ = Kripke frames (the logic is **K** based on language with ∇)
- proof theory for such logic (Hilbert style axiomatization, Gentzen style sequent calculi)

Creating calculi - case of the power set functor coalgebras

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  - should allow generalizations to a suitable class of functors

Outline

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- One-sided Gentzen calculi
    - a coalgebraic calculus
    - Janin Walukiewicz's tableau (95)

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  - a coalgebraic calculus
  - Janin Walukiewicz's tableau (95)
- Two-sided Gentzen calculus
  - One-sided Gentzen calculi are not available for negation-free languages.
  - Reflects better the underlying consequence relation

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$\mathcal{L}$	$a, b, c, \ldots$
$\mathcal{T}_\omega \mathcal{L}$	$\alpha, \beta, \gamma \ldots$
$\mathcal{P}_\omega \mathcal{L}$	$\varphi, \psi, \theta \ldots$
$\mathcal{T}_\omega \mathcal{P}_\omega \mathcal{L}$	$\Phi, \Psi, \Theta \ldots$
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$\mathcal{P}_\omega\mathcal{T}_\omega\mathcal{L}$	$A, B, C \ldots$

- ... to visualize where we depend on the fact $\mathcal{T}_\omega = \mathcal{P}_\omega$.

Cover modality

$$\mathbb{M}, s \Vdash \nabla\alpha$$

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iff

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$$R(s)\overline{\Vdash}\alpha$$

meaning that nabla distributes α over $R(s)$ so that $R(s)$ and (values of formulas from) α cover each other.

Redistributions - generalizing lifting

α_1				
$\vdots$	$\overline{\in}$	Φ		
α_m		$\Cup$		
	φ_1	$\dots$	φ_n	

Redistributions - generalizing lifting

$$
\boxed{\begin{array}{llll}
\alpha_1 & & & \\
\vdots & \overline{\in} & \Phi & \\
\alpha_m & & \Cup & \\
 & \varphi_1 & \cdots & \varphi_n
\end{array}}
$$

$\mathcal{TPL} \ni \Phi = \{\varphi_1, \ldots, \varphi_n\}$ is a redistribution of $\mathcal{PTL} \ni A = \{\alpha_1, \ldots, \alpha_m\}$ if each $\alpha_i \overline{\in} \Phi$. Such redistribution is slim if moreover $\bigcup \Phi = \bigcup A$.

Redistribution - examples

A classical distributive law:

$$\bigvee\{\bigwedge\alpha|\alpha\in A\}\equiv\bigwedge\{\bigvee\varphi|\varphi\overline{\in}A\}$$

where

$$\{\varphi|\varphi\overline{\in}A\}\in SRD(A)$$

Redistribution - examples

$$\mathbb{S}, s \Vdash \bigwedge\{\nabla\alpha \mid \alpha \in A\} \iff \{\{a \in \bigcup A \mid \mathbb{S}, t \Vdash a\} \mid t \in R(s)\} \in SRD(A).$$

An axiomatization

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(∇1)	From $\vdash \alpha \overline{\leq} \beta$ infer $\vdash \nabla\alpha \leq \nabla\beta$
(∇2)	$\bigwedge\{\nabla\alpha \mid \alpha \in A\} \leq \bigvee\left\{\nabla\{\bigwedge\varphi \mid \varphi \in \Phi\} \mid \Phi \in SRD(A)\right\}$
(∇3)	$\nabla\{\bigvee\varphi \mid \varphi \in \Phi\} \leq \bigvee\{\nabla\beta \mid \beta \,\overline{\in}\, \Phi\}$

One-sided Gentzen calculi - I Coalgebraic $G1\nabla$

sequents of the form $\varphi \Rightarrow \varnothing$, reading '$\bigwedge \varphi$ is not satisfiable'

One-sided Gentzen calculi - I Coalgebraic $G1\nabla$

$G1\nabla$ consists of the axiom scheme $p, \neg p \Rightarrow \varnothing$ and the rules:

One-sided Gentzen calculi - I Coalgebraic $G1\nabla$

$$\bigwedge\text{-I}\ \frac{\varphi, \psi \Rightarrow \varnothing}{\varphi, \bigwedge \psi \Rightarrow \varnothing} \qquad \bigvee\text{-I}\ \frac{\{\varphi, a \Rightarrow \varnothing | a \in \psi\}}{\varphi, \bigvee \psi \Rightarrow \varnothing} \qquad \text{weak-I}\ \frac{\varphi \Rightarrow \varnothing}{\varphi, a \Rightarrow \varnothing}$$

One-sided Gentzen calculi - I Coalgebraic $G1\nabla$

... what about a ∇ rule ???

$$\nabla\text{-}1 \; \frac{}{\{\nabla\alpha \mid \alpha \in A\} \Rightarrow \varnothing}$$

$$\nabla\text{-}1\ \frac{|\Phi \in SRD(A)\}}{\{\nabla\alpha|\alpha \in A\} \Rightarrow \varnothing}$$

$$\nabla\text{-}1 \ \frac{\{\varphi_\Phi \Rightarrow \varnothing \mid \Phi \in SRD(A)\}}{\{\nabla\alpha \mid \alpha \in A\} \Rightarrow \varnothing} \ \varphi_\Phi \in \Phi$$

$$\nabla\text{-}1 \; \frac{\{\varphi_\Phi \Rightarrow \varnothing | \Phi \in \mathit{SRD}(A)\}}{\{\nabla\alpha | \alpha \in A\} \Rightarrow \varnothing} \; \varphi_\Phi \in \Phi$$

$$\nabla\text{-}1 \; \frac{\{\varphi_\Phi \Rightarrow \varnothing \mid \Phi \in SRD(A)\}}{\{\nabla\alpha \mid \alpha \in A\} \Rightarrow \varnothing} \; \varphi_\Phi \in \Phi$$

... to be read as follows: Given A, if for every $\Phi \in SRD(A)$ there exists some $\varphi_\Phi \in \Phi$ such that $\varphi_\Phi \Rightarrow \varnothing$, then $\{\nabla\alpha \mid \alpha \in A\} \Rightarrow \varnothing$.

soundness, completeness

A distributive law behind the rule:

$$(\nabla 2): \bigwedge\{\nabla\alpha \mid \alpha \in A\} \leq \bigvee \{\nabla\{\bigwedge \varphi \mid \varphi \in \Phi\} \mid \Phi \in SRD(A)\}$$

soundness, completeness

The following are equivalent (ξ a collection of literals):

1. $\{\nabla\alpha \mid \alpha \in A\} \cup \xi$ is satisfiable.
2. ξ is satisfiable and for some $\Phi \in SRD(A)$, φ is satisfiable for every $\varphi \in \Phi$.

An example of a proof in $G1\nabla$

$$\nabla\text{-}1 \ \frac{\{\varphi_\Phi \Rightarrow \varnothing | \Phi \in \mathit{SRD}(A)\}}{\{\nabla\alpha | \alpha \in A\} \Rightarrow \varnothing} \ \varphi_\Phi \in \Phi$$

$$\frac{\bot \Rightarrow \varnothing}{\nabla\{\bot\} \Rightarrow \varnothing}$$

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where $A = \{\{\bot\}\}$, $\Phi = \{\{\bot\}\}$ is the only slim redistribution of A.

One-sided Gentzen calculi - II Janin and Walukiewicz's Tableau $GW\nabla$

Invented by Janin and Walukiewicz as a tableau to convert formulas of modal μ calculus to a disjunctive form

One-sided Gentzen calculi - II Janin and Walukiewicz's Tableau $GW\nabla$

given A and $a \in \bigcup A$:

$$\varphi_a = \{a\} \cup \{\bigvee \alpha' \mid \alpha' \in A \text{ and } a \notin \alpha'\}.$$

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The propositional fragment of Janin and Walukiewicz's tableaux $GW\nabla$: the axioms, the rules for boolean connectives and the weakening rule of $G1\nabla$, plus the following nabla-rule:

$$W\nabla \; \frac{\varphi_a \Rightarrow \varnothing}{\{\nabla\alpha | \alpha \in A\} \Rightarrow \varnothing} \; a \in \bigcup A$$

Relating the two calculi

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- both sound and complete, thus equivalent (derive the same sequents)
- the rules of one system, being sound, are admissible in the other system
- $G1\nabla$ simulates $GW\nabla$
- $GW\nabla$ based on the fact that $\mathcal{P}_\omega = \mathcal{T}_\omega$

$$\varphi_a = \{a\} \cup \{\bigvee \alpha' \mid \alpha' \in A \text{ and } a \notin \alpha'\}$$

Two-sided Gentzen calculus $G2\nabla$

for the positive fragment of the logic

Two-sided Gentzen calculus $G2\nabla$

sequents of the form $\varphi \Rightarrow \psi$, reading $\bigwedge \varphi \leq \bigvee \psi$

Two-sided Gentzen calculus *G*2∇

The sequent calculus *G*2∇: the axiom scheme $a \Rightarrow a$ and the rules:

$$\bigwedge\text{-l}\ \frac{\varphi, \theta \Rightarrow \psi}{\varphi, \bigwedge \theta \Rightarrow \psi} \qquad \bigvee\text{-r}\ \frac{\varphi \Rightarrow \theta, \psi}{\varphi \Rightarrow \bigvee \theta, \psi}$$

$$\bigwedge\text{-r}\ \frac{\{\varphi \Rightarrow a, \psi \mid a \in \theta\}}{\varphi \Rightarrow \bigwedge \theta, \psi} \qquad \bigvee\text{-l}\ \frac{\{\varphi, a \Rightarrow \psi \mid a \in \theta\}}{\varphi, \bigvee \theta \Rightarrow \psi}$$

$$\text{weak-r}\ \frac{\varphi \Rightarrow \psi}{\varphi \Rightarrow \psi, a} \qquad \text{weak-l}\ \frac{\varphi \Rightarrow \psi}{\varphi, a \Rightarrow \psi}$$

Two-sided Gentzen calculus $G2\nabla$

$$\text{cut}\ \frac{\varphi \Rightarrow \psi, a \qquad \varphi', a \Rightarrow \psi'}{\varphi, \varphi' \Rightarrow \psi, \psi'}$$

Two-sided Gentzen calculus $G2\nabla$

... the ∇ rule ???

$$\nabla \frac{\{\varphi \Rightarrow \theta | (\varphi, \theta) \in \qquad\qquad \}}{\{\nabla\alpha | \alpha \in A\} \Rightarrow \qquad\qquad \{\nabla\beta | \beta \in \qquad \}}$$

$$\nabla \frac{\{\varphi \Rightarrow \theta | (\varphi, \theta) \in \qquad\qquad\qquad \}}{\{\nabla\alpha | \alpha \in A\} \Rightarrow \bigcup_{\Phi \in SRD(A)} \{\nabla\beta | \beta \,\overline{\in}\, \Theta_\Phi\}}$$

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$$R \in \alpha \bowtie \beta \iff (\alpha, \beta) \in \overline{R}$$

$$\nabla \frac{\{\varphi \Rightarrow \theta | (\varphi, \theta) \in \bigcup_{\Phi \in SRD(A)} Y_\Phi\}}{\{\nabla \alpha | \alpha \in A\} \Rightarrow \bigcup_{\Phi \in SRD(A)} \{\nabla \beta | \beta \mathrel{\overline{\in}} \Theta_\Phi\}} \; Y_\Phi \in \Phi \bowtie \Theta_\Phi \text{ for all } \Phi \in SRD(A)$$

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... to be read as follows: Given $A \in \mathcal{P}_\omega \mathcal{T}_\omega \mathcal{L}$, if for every $\mathcal{T}_\omega \mathcal{P}_\omega \mathcal{L} \ni \Phi \in SRD(A)$ there exists some $\Theta_\Phi \in \mathcal{T}_\omega \mathcal{P}_\omega \mathcal{L}$ and some $Y_\Phi \in \Phi \bowtie \Theta_\Phi$ such that $\varphi \Rightarrow \theta$ for every $(\varphi, \theta) \in Y_\Phi$, then the conclusion follows.

An example of a proof

$$\nabla \frac{\{\varphi \Rightarrow \theta | (\varphi, \theta) \in \bigcup_{\Phi \in SRD(A)} Y_\Phi\}}{\{\nabla\alpha | \alpha \in A\} \Rightarrow \bigcup_{\Phi \in SRD(A)} \{\nabla\beta | \beta \mathrel{\overline{\in}} \Theta_\Phi\}} \; Y_\Phi \in \Phi \bowtie \Theta_\Phi \text{ for all } \Phi \in SRD(A)$$

$$\frac{a, b \Rightarrow a \qquad a, b \Rightarrow b}{\nabla\{a\}, \nabla\{b\} \Rightarrow \nabla\{a, b\}}$$

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$$\frac{a, b \Rightarrow a \qquad a, b \Rightarrow b}{\nabla\{a\}, \nabla\{b\} \Rightarrow \nabla\{a, b\}}$$

Here $A = \{\{a\}, \{b\}\}$ and $\Phi = \{\{a, b\}\}$ is the only slim redistribution of A. $\Theta_\Phi = \{\{a\}, \{b\}\}$, and $\beta = \{a, b\}$ is the only $\beta \,\overline{\in}\, \Theta_\Phi$.

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$$\frac{\varnothing \Rightarrow \top \qquad \varnothing}{\varnothing \Rightarrow \nabla\varnothing, \nabla\{\top\}}$$

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$$\frac{\varnothing \Rightarrow \top \qquad \varnothing}{\varnothing \Rightarrow \nabla\varnothing, \nabla\{\top\}}$$

where $A = \varnothing$, $\Phi_1 = \{\varnothing\}$ and $\Phi_2 = \varnothing$ are the only slim redistributions of A, $\Theta_{\Phi_1} = \{\{\top\}\}$ and $\Theta_{\Phi_2} = \varnothing$, and $\beta_1 = \{\top\}$ and $\beta_2 = \varnothing$.

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a counterexample:

$$\nabla\{p \vee q\} \Rightarrow \nabla\{p, \top\}, \nabla\{q\}$$

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a counterexample:

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is provable in $G2\nabla$ but not in the system obtained from $G2\nabla$ by removing the cut rule.