Counterpart Semantics at work: An Incompleteness Result in Quantified Modal Logic

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Main Result

Any First-order Extension of Normal Propositional Modal Logic obtained by adding Free Logic's Theory of Quantification and BF is Kripke-incomplete.

Motivations I

The motivation for this talk comes from an interest in philosophical logic.

- In [Kri63] Kripke introduced a QML system based on a non-classical theory of quantification to provide a formal account of Kripke frames with varying domains.

Motivations II

We lack a general framework for proving Kripke-completeness of QML systems based on classical and non-classical first-order logic:

- In [Cor02] Corsi tries to provide a completeness proof for QML systems based on classical and free logic, and Kripke's theory of quantification.

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- In [Gar05] Garson considers completeness for Kripke structures with domains of intensional objects.
- [Cor05, Gar05]: the systems $Q^{\circ}.B+BF$ and $Q^{\circ}.S5+BF$ are Kripke-incomplete.
- [this talk]: the systems $Q^{E}.K+BF$ and $Q^{E}.K+CBF+BF$ are Kripke-incomplete.

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The idea is rather simple:

- Every Kripke model of Q^E.K+BF is also a model of the necessity of fictionality, N¬E: $\neg E(x) \rightarrow \Box \neg E(x)$.

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- However, Q^E.K+BF does not prove N¬E.
- in [Gar05] Garson proves the independence of N¬E from his system GBF.
- Q^E.K+BF+N¬E is complete w.r.t. Kripke frames with decreasing inner domains and constant outer domains.

The Systems Q^E.K+BF and Q^E.K+CBF+BF

- Definition (Language $\mathcal{L}$)

$$\phi \quad ::= \quad P^n(x_1, \ldots, x_n) \mid E(x) \mid \neg\phi \mid \phi \rightarrow \psi \mid \forall x \phi \mid \Box\phi$$

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- Definition (System Q^E.K+BF)

Taut	tautologies of classical propositional calculus
K	$\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$
MP	$\phi \to \psi, \phi \Rightarrow \psi$
Nec	$\phi \Rightarrow \Box\phi$
E-Ex	$\forall x\phi \to (E(y) \to \phi[x/y])$
E-Gen	$\phi \to (E(x) \to \psi) \Rightarrow \phi \to \forall x\psi$, x is not free in ϕ
BF	$\forall x\Box\phi \to \Box\forall x\phi$

Q^E.K+CBF+BF extends Q^E.K+BF by adding CBF: $\Box\forall x\phi \to \forall x\Box\phi$

Kripke Frames and Models

- Definition (K-frame)

  A Kripke frame is a tuple $\mathcal{F} = \langle W, R, D, d \rangle$ such that

  - $W \neq \emptyset$ and $R \subseteq W^2$
  - for $w, w' \in W$, $D(w) \neq \emptyset$ and $wRw' \Rightarrow D(w) \subseteq D(w')$
  - for $w \in W$, $d(w) \subseteq D(w)$

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- Definition (K-model)

  A Kripke model is a pair $\mathcal{M} = \langle \mathcal{F}, I \rangle$ where I is an interpretation of $\mathcal{L}$ s.t.
  - $I(P^n, w) \subseteq (D(w))^n$ and $I(E, w) = d(w)$

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- **Definition (Satisfaction $\models$)**
  For $w \in \mathcal{M}$, $\phi \in \mathcal{L}$, and a w-assignment $\sigma : Var \rightarrow D(w)$:

$$
\begin{array}{lll}
(\mathcal{M}^\sigma, w) \models P^n(x_1, \ldots, x_n) & \text{iff} & \langle \sigma(x_1), \ldots, \sigma(x_n) \rangle \in I(P^n, w) \\
(\mathcal{M}^\sigma, w) \models \neg\psi & \text{iff} & (\mathcal{M}^\sigma, w) \not\models \psi \\
(\mathcal{M}^\sigma, w) \models \psi \rightarrow \psi' & \text{iff} & (\mathcal{M}^\sigma, w) \not\models \psi \text{ or } (\mathcal{M}^\sigma, w) \models \psi' \\
(\mathcal{M}^\sigma, w) \models \Box\psi & \text{iff} & \text{for } w' \in W,\ wRw' \Rightarrow (\mathcal{M}^\sigma, w') \models \psi \\
(\mathcal{M}^\sigma, w) \models \forall x\psi & \text{iff} & \text{for } a \in d(w),\ (\mathcal{M}^{\sigma\binom{x}{a}}, w) \models \psi
\end{array}
$$

Frames and Validities

- Remark

  For every K-frame $\mathcal{F}$,

$$\begin{array}{lll} \mathcal{F} \models Q^E.K{+}BF & \text{iff} & wRw' \Rightarrow d(w') \subseteq d(w) \\ \mathcal{F} \models Q^E.K{+}CBF{+}BF & \text{iff} & wRw' \Rightarrow d(w') = d(w) \end{array}$$

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- Corollary

$$\begin{array}{lll} Q^E.K{+}BF & \models & N\neg E \\ Q^E.K{+}CBF{+}BF & \models & N\neg E \end{array}$$

Counterpart Frames and Models

- Definition (c-frame)

  A counterpart frame is a tuple $\mathcal{G} = \langle W, R, D, d, C \rangle$ such that

  - $W \neq \emptyset$ and $R \subseteq W^2$
  - for $w \in W$, $D(w) \neq \emptyset$ and $d(w) \subseteq D(w)$
  - for wRw', $C_{w,w'} \subseteq D(w) \times D(w')$

existentially faithful	iff	wRw' & $a \in d(w)$ & $C_{w,w'}(a, b) \Rightarrow b \in d(w')$
fictionally faithful	iff	wRw' & $a \in D(w) \setminus d(w)$ & $C_{w,w'}(a, b) \Rightarrow b \in D(w') \setminus d(w')$
total	iff	wRw' & $a \in D(w) \Rightarrow$ there is $b \in D(w')$ s.t. $C_{w,w'}(a, b)$
surjective	iff	wRw' & $b \in d(w') \Rightarrow$ there is $a \in d(w)$ s.t. $C_{w,w'}(a, b)$
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- Definition (c-model)

  A counterpart model is a couple $\mathcal{N} = \langle \mathcal{G}, I \rangle$ where I is defined as in Def. 4.

The Need of Types

The following instance of axiom K is not valid:

$$\Box(Q(x_1, x_2) \to D(x_2)) \to (\Box Q(x_1, x_2) \to \Box D(x_2))$$

Consider a c-model $\mathcal{N}$ where:

- $W = \{w, w'\}$ and $R = \{(w, w')\}$
- $D(w) = d(w) = \{a, b\}$ and $D(w') = d(w') = \{b\}$
- $C_{w,w'} = \{(b, b)\}$

Further, $I(D, w') = \emptyset$, $\sigma(x_1) = a$ and $\sigma(x_2) = b$

Consider the following definition of satisfaction:

$(\mathcal{N}^\sigma, w) \models \Box\phi[x_1, \ldots, x_n]$ iff for every $w' \in W$, for every w'-assignment τ, wRw' and $C_{w,w'}(\sigma(x_i), \tau(x_i))$ imply $(\mathcal{N}^\tau, w') \models \phi[x_1, \ldots, x_n]$

$$\begin{array}{lcl} (\mathcal{N}^\sigma, w) & \models & \Box(Q(x_1, x_2) \to D(x_2)) \wedge \Box Q(x_1, x_2) \\ (\mathcal{N}^\sigma, w) & \not\models & \Box D(x_2) \end{array}$$

The Typed First-order Modal Language $\mathcal{L}_T$

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- Definition (Language $\mathcal{L}_T$)
  - if $t_1, \ldots, t_m : n$, then $P^m(t_1, \ldots, t_m) : n$;
  - if $\psi : n$ and $\psi' : n$, then $\neg\psi : n$ and $\psi \to \psi' : n$;
  - if $\psi : m$ and $t_1, \ldots, t_m : n$, then $(\Box\psi)(t_1, \ldots, t_m) : n$
  - if $\psi : n+1$, then $\forall x_{n+1}\psi : n$

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- Definition (Satisfaction $\models$)

  For $w \in \mathcal{N}$, an n-formula ϕ, and an n-tuple $\vec{a}$:

$$(\mathcal{N}^{\vec{a}}, w) \models (\Box\psi)(t_1, \ldots, t_m) \text{ iff for } w' \in W, \text{ for } b_1, \ldots, b_m \in D(w'), \; wRw' \text{ and } C_{w,w'}(\vec{a}(t_i), b_i) \text{ imply } (\mathcal{N}^{\vec{b}}, w') \models \psi$$

Validities

- substitution does not commute with the modal operator:

$$\begin{array}{ll} \models & (\Box\phi)[t_1, \ldots, t_m] \to \Box(\phi[t_1, \ldots, t_m]) \\ \not\models & \Box(\phi[t_1, \ldots, t_m]) \to (\Box\phi)[t_1, \ldots, t_m] \end{array}$$

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where $\Box\phi = (\Box\phi)(x_1, \ldots, x_n) : n$

- Remark

For every c-frame $\mathcal{G}$,

$$\begin{array}{lll} \mathcal{G} \models BF_T & \textit{iff} & \mathcal{G} \textit{ is surjective} \\ \mathcal{G} \models N\neg E_T & \textit{iff} & \mathcal{G} \textit{ is fictionally faithful} \end{array}$$

There is a surjective c-frame $\mathcal{G}$ such that $\mathcal{G} \models \mathrm{BF}_T$, but $\mathcal{G} \not\models \mathrm{N}\neg\mathrm{E}_T$.

Incompleteness of QML Systems

Theorem

The system $Q^E.K{+}BF$ is Kripke-incomplete, i.e., $Q^E.K{+}BF \models N\neg E$, but $Q^E.K{+}BF \nvdash N\neg E$.

- If $Q^E.K{+}BF \vdash \phi$, then every total, surjective and functional c-frame $\mathcal{G} \models \tau_n(\phi)$.

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- If $Q^E.K{+}BF \vdash \phi$, then every total, surjective and functional c-frame $\mathcal{G} \models \tau_n(\phi)$.
- There is a total, surjective and functional c-frame $\mathcal{G}$ such that $\mathcal{G} \not\models \tau_n(N\neg E)$.

Incompleteness of QML Systems

Following [Cor05, Ghi90] we define a translation function $\tau : \mathcal{L} \rightarrow \mathcal{L}_T$.

Definition

Let $\phi \in \mathcal{L}$ and define $g(\phi)$ as the maximum k such that x_k occurs in ϕ.
For $n \geq g(\phi)$, $\tau_n(\phi) : n$ in $\mathcal{L}_T$ is defined as follows:

$$
\begin{array}{lcl}
\tau_n(P^m(t_1, \ldots, t_m)) & := & P^m(t_1, \ldots, t_m) \\
\tau_n(\neg\psi) & := & \neg\tau_n(\psi) \\
\tau_n(\Box\psi) & := & \Box\tau_n(\psi) \\
\tau_n(\psi \rightarrow \psi') & := & \tau_n(\psi) \rightarrow \tau_n(\psi') \\
\tau_n(\forall x_i \psi) & := & \forall x_{n+1}(\tau_n(\psi)[x_1, \ldots, x_{i-1}, x_{n+1}, x_{i+1}, \ldots, x_n])
\end{array}
$$

Incompleteness of QML Systems

- Lemma
  Let $\phi \in \mathcal{L}$, $n \geq g(\phi)$ and let $\mathcal{G}$ be a total, surjective, and functional c-frame, then

$$Q^E.K + BF \vdash \phi \quad \Rightarrow \quad \mathcal{G} \models \tau_n(\phi)$$

The proof of this lemma requires the following auxiliary result, in which the assumptions of everywhere-definiteness and functionality are essential.

Incompleteness of QML Systems

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$$Q^E.K + BF \vdash \phi \quad \Rightarrow \quad \mathcal{G} \models \tau_n(\phi)$$

  The proof of this lemma requires the following auxiliary result, in which the assumptions of everywhere-definiteness and functionality are essential.

- Lemma
  If $\phi \in \mathcal{L}$, $\mathcal{G}$ is a total and functional c-frame, and $x_{i_1}, \ldots, x_{i_m}$ are free for $x_1, \ldots, x_m$ in ϕ, then

$$\mathcal{G} \models \tau_m(\phi)[x_{i_1}, \ldots, x_{i_m}] \leftrightarrow \tau_n(\phi[x_{i_1}, \ldots, x_{i_m}])$$

  Under the assumptions of everywhere-definiteness and functionality, substitution commutes with the modal operator.

Incompleteness of QML Systems

Lemma

There is a total, surjective and functional c-frame $\mathcal{G}$ such that $\mathcal{G} \not\models \tau_n(N\neg E)$.

Consider the c-frame $\mathcal{G}$ where

- $W = \{w, w'\}$ and $R = \{(w, w')\}$;
- $D(w) = \{a, a'\}$, $D(w') = \{b\}$;
- $d(w) = \{a\}$, $d(w') = \{b\}$;
- $C_{w,w'} = \{(a, b), (a', b)\}$.

By definition $\mathcal{G}$ is total, surjective and functional.
But N¬E fails in $\mathcal{G}$ as it is not fictionally faithful.

Incompleteness of QML Systems

Theorem

The system $Q^E.K{+}CBF{+}BF$ is Kripke-incomplete, i.e., $Q^E.K{+}CBF{+}BF \models N\neg E$ but $Q^E.K{+}CBF{+}BF \nvdash N\neg E$.

- The proof goes as for $Q^E.K{+}BF$, but we consider total, surjective and functional c-frames, which are also existentially faithful.

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- The proof goes as for $Q^E.K{+}BF$, but we consider total, surjective and functional c-frames, which are also existentially faithful.
- Note that the c-frame in the previous lemma is also existentially faithful.

Discussion

Modalities stronger than K:

- The incompleteness result extends to QML calculi on modal bases T and $S4$, but not to modal bases B and $S5$.

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Discussion

Modalities stronger than K:

- The incompleteness result extends to QML calculi on modal bases T and $S4$, but not to modal bases B and $S5$.
- In [Gar05] an intensional semantics is introduced, capable of dealing with all normal modalities.
- Also the systems Q^E.B+BF and Q^E.S5+BF are Kripke-incomplete.

Open Problems

Open problems concerning the completeness of non-classical QML systems:

- $Q^\circ.K+BF$ and $Q^\circ.K+CBF+BF$

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- $Q^{\circ}.K+BF$ and $Q^{\circ}.K+CBF+BF$
- $Q^{\circ}.B$ and $Q^{\circ}.S5$

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Open problems concerning the completeness of non-classical QML systems:

- $Q^{\circ}.K{+}BF$ and $Q^{\circ}.K{+}CBF{+}BF$
- $Q^{\circ}.B$ and $Q^{\circ}.S5$
- $Q^{\circ}.B{+}CBF$ and $Q^{\circ}.S5{+}CBF$

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